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ON THE INDEPENDENCE OF THE AXIOM OF CONSTRUCTIBILITY.*[1]

By J. R. SHOENFIELD.

In [2], Gödel has shown that the Axiom of Choice and the Generalized Continuum Hypothesis are conseqences of a more general axiom. This axiom, which we call the Axiom of Constructibility, states that every set is constructible (in the notation of [2], $V = L$).

It is natural to ask if the Axiom of Constructibility follows from the Axiom of Choice and the Generalized Continuum Hypothesis. A difficulty in answering this question is that the independence of the Axiom of Constructibility from the axioms of set theory (i.e., the axioms $A - D$ of [2]) has not been settled. This suggests we try to prove the following relative independence statement: if the Axiom of Constructibility is not provable from axioms $A - D$, then it is not provable from axioms $A - D$, the Axiom of Choice, and the Generalized Continuum Hypothesis.

The object of this paper is to prove two results which are slightly weaker than the statement just mentioned.[2] We first introduce weakened forms of the Generalized Continuum Hypothesis and the Axiom of Constructibility.

GCH′: *There is an ordinal* α_0 *such that* $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ *for* $\alpha \geqq \alpha_0$.

ACon′: *Every set of integers is constructible.*

THEOREM 1. *If the Axiom of Constructibility is not provable from axioms* $A - D$, *then it is not provable from axioms* $A - D$, *the Axiom of Choice, and GCH′.*

THEOREM 2. *If ACon′ is not provable from axioms* $A - C$, *then it is*

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* Received March 17, 1958.

[1] Presented to the Summer Institute of Symbolic Logic in 1957 at Cornell University. This research was supported by a National Science Foundation fellowship.

[2] The possibility of proving Theorem 2 by the methods of this article was noted some time ago by Gödel.

The results of this article have been obtained independently by Lévy. More recently, Lévy has obtained extensions of these results, which will be published soon.

not provable from axioms $A-D$, the Axiom of Choice, and the Generalized Continuum Hypothesis (and hence the Axiom of Constructibility is not provable from these axioms).

We now give the proof of Theorem 1. Assume the hypothesis of the theorem. Then axioms $A-D$ and the axiom $V \neq L$ form a consistent set. From axiom D and $V \neq L$, it follows that there is a set $x \in V-L$ such that $x \cap (V-L) = 0$, i.e., such that $x \subset L$. Hence it is consistent with axioms $A-D$ to introduce a new individual constant 'a' and the axioms:

$$\mathfrak{M}(a) \tag{1}$$

$$a \notin L \tag{2}$$

$$a \subset L. \tag{3}$$

We now introduce the set operations $\mathfrak{F}_1-\mathfrak{F}_8$ of 9.1 (all decimal references are to [2]), and a ninth operation:

$$\mathfrak{F}_9(XY) = a \cap X.$$

We then introduce all the definitions of Chapter V of [2], with $\mathfrak{F}_1-\mathfrak{F}_8$ replaced by $\mathfrak{F}_1-\mathfrak{F}_9$. (In particular, we replace '9' by '10' in 9.2 and 9.21.) We designate the so defined concepts by the symbols used in [2] followed by an asterisk; e.g., L^*, constructible*.

Now we introduce a model Δ^* as follows: class is interpreted as constructible* class; set is interpreted as constructible* set; membership is interpreted as membership; and a is interpreted as a.

The results of Chapters 5-7 of [2] hold also for the model Δ^*. For the only place in which the operations $\mathfrak{F}_1-\mathfrak{F}_8$ are considered individually is in the proof of 9.5; and this is easily extended to include $\mathfrak{F}_9$, since $\mathfrak{F}_9(xy) \subset x$. Hence we obtain: Δ^* is a model for axioms $A-D$, and L is absolute*.

If we relativize the theorem $L \subset V$ to the model Δ^*, we obtain $L \subset L^*$. Hence by (3), $a \subset L^*$. By the analogue of 9.63, it follows that $a \subset x$ for some $x \in L^*$. Since $a = \mathfrak{F}_9(x, x)$, we have $a \in L^*$, which is the relativization of (1) to Δ^*. Since (2) and (3) are absolute*, Δ^* is a model for (1)-(3). Noting that $\mathfrak{F}_9$ is absolute*, we prove as in Chapter 7 of [2] that L^* is absolute*, and hence that the relativization of $V = L^*$ to Δ^* is $L^* = L^*$. Hence Δ^* is a model for $V = L^*$. This proves:

LEMMA. *If $V = L$ is not provable from axioms $A-D$, the axioms $A-D$, (1)-(3), and $V = L^*$ are consistent.*

Theorem 1 will follow from the lemma if we show $A - D$, (1)-(3), and $V = L^*$ imply the Axiom of Choice and GCH′. For the Axiom of Choice, this is immediate (see [2], page 53). To prove GCH′, choose α_0 so that the order* of every element of a is $< \aleph_{\alpha_0}$. We then proceed as in Chapter VIII of [2]. The proofs of 12.4, 12.5, and 12.51 are unchanged. We modify 12.6 by adding the hypothesis that m and m' include $\aleph_{\alpha_0}$. In the proof of 12.6, it is necessary to add a further case to the cases on page 60. The result needed is:

$$F^* `\alpha \in a \cap F^* `\beta \equiv F^* `\alpha' \in a \cap F^* `\beta'.$$

In view of the induction hypothesis I, this will follow from:

$$F^* `\alpha \in a \equiv F^* `\alpha' \in a.$$

By symmetry, it is sufficient to show that $F^* `\alpha \in a$ implies $F^* `\alpha' \in a$. Let γ be the order* of $F^* `\alpha$. Then $\gamma < \aleph_{\alpha_0}$. Since $m$ and $m'$ include $\aleph_{\alpha_0}$ and $G$ is an order isomorphism, $\gamma' = \gamma$. Hence, using induction hypothesis II and noting $\gamma \leqq \alpha < \eta$, we have $F^* `\alpha = F^* `\gamma = F^* `\gamma' = F^* `\alpha'$; whence $F^* `\alpha' \in a$.

We then prove as in [2] that 12.6 implies 12.3, provided that in 12.3 we require m and o to include $\aleph_{\alpha_0}$. This form of 12.3 implies 12.2 whenever $\alpha \geqq \alpha_0$. For in the proof of 12.2 from 12.3, m is chosen so that $\omega_{\alpha_0} \subset \omega_\alpha \subset m$; and o is an ordinal such that $\overline{\overline{o}} = \omega_\alpha$, so that $\omega_{\alpha_0} \subset \omega_\alpha \subset o$. Finally, 12.2 for $\alpha \geqq \alpha_0$ implies GCH′ as in [2]. This completes the proof of Theorem 1.

Theorem 2 is proved similarly. We first observe that axiom D is not required in the consistency proof of [2]. (The uses of axiom D in proving M1 on page 10 are easily eliminated. The development of ordinals without use of axiom D is explained, e.g., in [3]).

We now proceed as in the proof of Theorem 1. From the consistency of the negation of ACon′ with $A - D$, we obtain the consistency of (1), (2), and:

(3′) $$a \subset \omega.$$

Clearly (3′) implies (3). In the proof of the lemma, we need only make the additional observation that (3′) is absolute*. (This is proved by the methods of Chapter VII of [2].) In the last part of the proof, we need only observe that we may take $\alpha_0 = 0$; for, as is easily proved, the order* of every integer is finite.

It remains to prove $V = L^*$ implies axiom D. Let A be a non-empty

constructible* class, and let x be the element of A having the smallest order*. Then $\mathfrak{Ex}(x, A)$ by 9.5.

We note that axiom D apparently cannot be omitted from the hypothesis of Theorem 1. For since axiom D is known to be independent of axioms $A - C$ ([1]), and since $V = L$ implies axiom D (as above), $V = L$ is independent of axioms $A - C$.

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IMAGES OF PLANE CONTINUA.*

By M. K. Fort, Jr.[1]

1. Introduction. A *continuum* is a compact, connected, metric space. We define K to be the continuum which consists of all points in the plane having polar coordinates (r, θ) for which $r = 1$, $r = 2$, or $r = (2 + e^{\theta})/(1 + e^{\theta})$.

The following two theorems are the main results obtained in this paper.

Theorem 1. *A plane continuum which does not separate the plane cannot be mapped continuously onto K.*

Theorem 2. *A plane continuum cannot be mapped continuously onto the dyadic solenoid.*

The second of these theorems generalizes the known theorem which states that the dyadic solenoid cannot be imbedded in a plane.

Our two theorems give information pertinent to the following questions, whose answers are not known, and which seem to be of interest to topologists.[2]

Question 1. Is there a continuum which can be mapped continuously onto every other continuum?

Question 2. Is there a plane continuum which can be mapped continuously onto every other plane continuum?

Question 3. What characterizes all continuous images of a pseudo-arc?

Question 4. What characterizes all plane continuous images if a pseudo-arc?

It follows from these theorems that:

(i) if the continuum sought in Question 1 exists, it cannot be a plane continuum;

* Received June 27, 1958.

[1] This research was partially supported by National Science Foundation Research Grant NSF-G3016.

[2] Questions 1 and 2 were raised by R. H. Bing during a recent conversation with the author. Question 3 is stated in [1], p. 72.

(ii) if the continuum sought in Question 2 exists, it must separate the plane; and

(iii) not every plane continuum is the continuous image of the pseudo-arc.

2. Proof of Theorem 1. The following lemma concerning locally trivial fiber spaces (see [1], p. 111, and [2]) is used both in the proof of Theorem 1 and in the proof of Theorem 2.

LEMMA. *Let (E, B, p) be a locally trivial fiber space such that for each $b \in B$, the fibre $p^{-1}(b)$ is totally disconnected. If f is a mapping of a connected space A into E such that pf is homotopic to a constant, then $f[A]$ is contained in a single arc component of E.*

Proof. Let $h: A \times I \to B$ be a homotopy such that $h(x, 0) = pf(x)$ and $h(x, 1) = b$ for some $b \in B$ and each $x \in A$. Because of the covering homotopy property, there is a homotopy $g: A \times I \to E$ such that $g(x, 0) = f(x)$ and $pg = h$. Now $g \mid A \times \{1\}$ is a mapping of a connected space into the totally disconnected set $p^{-1}(b)$. It follows that there exists $q \in p^{-1}(b)$ such that $g(x, 1) = q$ for all $x \in A$. Now let (x, t) be any point of $A \times I$. The mapping g takes the segment $\{(x, s) \mid t \leqq s \leqq 1\}$ into a locally connected continuum containing q, and hence $g(x, t)$ belongs to the same arc component of E as q. It follows that $f[A]$ is contained in the same arc component of E as q.

Proof of Theorem 1. We let S_1 be the unit circle in the plane, and define $p: K \to S_1$ by $p(r, \theta) = (1, \theta)$. It is easy to verify that (K, S_1, p) is a locally trivial fiber space with totally disconnected fibers.

Now, if A is a plane continuum which does not separate the plane, and f is a mapping of A into K, then pf is a mapping of a continuum which does not separate the plane into S_1. It is known (see [3], p. 93) that pf is homotopic to a constant. It follows from our lemma that $f[A]$ is contained in a single arc component of K. Our theorem now follows at once from the fact that K has three arc components.

3. Continuous roots of mappings. The proof which we give for Theorem 2 makes use of some results concerning continuous roots of mappings. We prove the facts which are needed in this section.

The unit circle S_1 is considered as the multiplicative group of complex numbers of unit modulus. If $f: A \to S_1$ is a mapping (i.e. continuous function), then a mapping $g: A \to S_1$ is a *continuous k-th root of f*, k a positive

integer, if $g(x)^k = f(x)$ for all $x \in A$. We say that f *has property D* if f has continuous 2^n-th roots for every positive integer n.

RESULT 1. *If f is homotopic to a constant, then f has property D.*

Proof. It is known (see [3], p. 68) that if f is homotopic to a constant, then there exists a continuous real valued function h on the domain A of f such that $f(x) = e^{ih(x)}$ for all $x \in A$. It follows that $g_k(x) = e^{ih(x)/k}$ defines a continuous k-th root g_k of f for each positive integer k. Thus f has property D.

RESULT 2. *If A is a locally connected continuum and $f: A \to S_1$ has property D, then f is homotopic to a constant.*

Proof. Since f is uniformly continuous, there exists $\epsilon > 0$ such that any subset of A of diameter less than ϵ maps under f into a subset of S_1 which is contained in an arc of length less than 1. Since A is a locally connected continuum, A has property S and we may express A as the union of subcontinua $A_1, \cdots, A_k$ each having diameter less than ϵ. Let f_j be a continuous 2^j-th root of f for each positive integer j. It is easy to see that each $f_j[A_m]$ is an arc of S_1 having length less than 2^{-j}. It follows that for $2^j > k$, $f_j[A]$ is a proper subset of S_1 (in fact, an arc of length less than $k2^{-j}$). Thus, we see that for large values of j, f_j is homotopic to a constant. Since f is a power of f_j for each j, f_j is also homotopic to a constant.

RESULT 3. *If U is an open subset of the plane, A is a compact subset of U, F is a mapping on U into S_1, and f is a continuous square root of $F \mid A$, then there exists an open set V such that $A \subset V \subset U$ and an extension of f to a continuous square root of $F \mid V$.*

Proof. Choose an open set W containing A such that the closure of W is compact and is contained in U. There exists $\delta_1 > 0$ such that if x and y are in W and $|x - y| < \delta_1$, then $|F(x) - F(y)| < 1/2$. There exists $\delta_2 > 0$ such that if a and b are in A and $|a - b| < \delta_2$, then $|f(a) - f(b)| < 1/2$. Let $\delta > 0$ be chosen so that $\delta < \delta_1$, $\delta < \delta_2$, and the $(\delta/2)$-neighborhood V of A is contained in W.

Now suppose $x \in V$. We choose $a \in A$ such that $|x - a| < \delta/2$. $F(x)$ has a unique square root s which is nearest to $f(a)$. We define $g(x) = s$. It is easy to verify that $g(x)$ is independent of the particular $a \in A$ chosen, that the function g is a continuous square root of $F \mid V$, and that g is an extension of f.

RESULT 4. *If $F: A \to B$, $G: B \to S_1$ and g is a continuous square root of G, then gF is a continuous square root of GF.*

Proof. Obvious.

RESULT 5. *If f and g are mappings of a space X into S_1 and f is homotopic to g, then f has a continuous square root if and only if g does.*

Proof. Let $p(z) = z^2$ for $z \in S_1$ and apply the homotopy lifting theorem for the locally trivial fiber space (S_1, S_1, p).

RESULT 6. *Let f be a mapping of a space X into S_1, and let A be a deformation retract of X. If $f \mid A$ has a continuous square root, then so does f.*

Proof. There exists $R: X \times I \to X$ such that:

$$R(x, 0) = x \text{ for all } x \in X,$$

$$R(x, 1) \in A \text{ for all } x \in X, \text{ and}$$

$$R(x, 1) = x \text{ for all } x \in A.$$

We define $G = fR$, and let $g_0(x) = G(x, 0)$, $g_1(x) = G(x, 1)$. It follows from Result 4 and the hypothesis that $f \mid A$ has a continuous square root that g_1 has a continuous square root. Therefore, by Result 5, g_0 has a continuous square root. Since $f = g_0$, we obtain the desired result.

RESULT 7. *If A is a topological annulus, $f: A \to S_1$, and f has a continuous square root on one of the boundary curves of A, then f has a continuous square root on A.*

Proof. This result is an immediate corollary of Result 6.

RESULT 8. *If C is a plane continuum and $f: C \to S_1$ has property D, then f is homomotopic to a constant.*

Proof. It is known that f is homotopic to a constant if f can be extended over the entire plane P. Any mapping of C into S_1 can be extended to a mapping of $P - A$ into S_1, where A is a finite set and is contained in the union of the bounded components of $P - C$. It follows that it is sufficient to prove that f can be extended to a mapping of $C \cup U$ into S_1 for each bounded component U of $P - C$.

Let U be a bounded component of $P - C$, and let E be an open disk whose closure is contained in U. It is known that f can be extended to a mapping g whose domain is open and contains $C \cup (U - E)$.

Since f, and hence also g, has a continuous square root on C, by Result 3

we see that g has a continuous square root ψ on some open set $V_1 \supset C$. There exists a closed topological disk F such that: $F \subset U$, the closure of E is contained in the interior of F, $F \cup V_1 \supset U$, and the boundary of F is contained in V_1. Then $F - E$ is a topological annulus, g is defined over $F - E$, and g has the continuous square root ψ on the outer boundary B of $F - E$. It follows from Result 7 that $\psi \mid B$ can be extended to a continuous square root ϕ of $g \mid (F - E)$. We can piece together ϕ and a part of ψ to obtain a continuous square root g_1 of g on $V_1 \cup (U - E)$.

We repeat the above process and obtain a continuous square root g_2 of g_1 on $V_2 \cup (U - E)$, where V_2 is open and $C \subset V_2 \subset V_1$. Continuing, we see that g has property D on the boundary B' of E. Thus, by Result 2, $g \mid B'$ is homotopic to a constant. Hence, $g \mid B'$ can be extended continuously over $B' \cup E$. We are now able to construct a continuous extension of f over $C \cup U$.

RESULT 9. *If C is the dyadic solenoid and F is a mapping of C into S_1, then F has a continuous square root.*

Proof. We represent C as an inverse limit space, $C = \varprojlim_n \{X_n, f_n\}$, where for each positive integer n, $X_n = S_1$, and $f_n : X_{n+1} \to X_n$ is defined by $f_n(z) = z^2$. we let π_n be the projection of C onto X_n for each positive integer n. The space C is metrizable, and we assume that C is metrized by the admissible metric d defined by

$$d(x, y) = \sum_{n=1}^{\infty} |\pi_n(x) - \pi_n(y)| / 2^n.$$

It is easily seen that if $u \in X_n$, then

$$\text{diameter } \pi_n^{-1}(u) \leqq 2^{-n}.$$

Now, for some integer k, we are going to obtain a mapping $g_k : X_k \to S_1$ such that $g_k \pi_k$ is homotopic to F. In order to accomplish this, we choose $\delta > 0$ such that if $B \subset C$ and diameter $B < \delta$, then diameter $F[B] < 1$. We choose k to be a positive integer for which $2^{-k} < \delta$. Since π_k^{-1} is an upper semi-continuous set-valued function and diameter $\pi_k^{-1}(u) \leqq 2^{-k}$ for each $u \in X_k$, there exists $\eta > 0$ such that if $A \subset X_k$ and diameter $A < \eta$ then diameter $\pi_k^{-1}[A] < \delta$. Next, we choose points $p_0, p_1, \cdots, p_m, p_0$ in cyclic order on X_k such that $|p_i - p_{i+1}| < \eta$ for each i. We define $g_k : X_k \to S_1$ such that:

$$g_k(p_i) \in F[\pi_k^{-1}(p_i)] \text{ for each } i,$$

and

$$g \text{ is "linear" on each arc } \overline{p_i p_{i+1}}.$$

For such a g_k, it follows that $|g_k\pi_k(x) - F(x)| < 1$ for all $x \in C$, and hence $g_k\pi_k$ is homotopic to F.

Now, since $\pi_k = f_k\pi_{k+1}$, we see that

$$g_k\pi_k = g_k(f_k\pi_{k+1}) = (g_kf_k)\pi_{k+1}.$$

Both g_k and f_k are mappings of S_1 into S_1, and f_k is of degree 2. Since

$$\text{degree}(g_kf_k) = (\text{degree } g_k)(\text{degree } f_k),$$

it follows that g_kf_k is of even degree and hence has a continuous square root. We now use Result 4 to see that $(g_kf_k)\pi_{k+1}$ has a continuous square root. Since $(g_kf_k)\pi_{k+1} = g_k\pi_k$ and $g_k\pi_k$ is homotopic to F, it follows from Result 5 that F has a continuous square root.

4. Proof of Theorem 2. We now use the results obtained in the preceding paragraph to prove Theorem 2.

Proof of Theorem 2. Let K be a plane continuum, and let G be a mapping of K into the dyadic solenoid C. We will show that $G[K]$ is contained in a single arc component of C, and hence, since C is not arcwise connected, G does not map K *onto* C.

We represent $C = \varprojlim_n \{X_n, f_n\}$, as in the proof of Result 9, and we again let π_1 be the projection of C onto $S_1 = X_1$. It is easily seen that (C, S_1, π_1) is a locally trivial fiber space with totally disconnected fibers.

It follows from Result 9 that π_1 has property D. Thus, by Result 4, π_1G has property D. Since π_1G is a mapping of a plane continuum into S_1, we apply Result 8 and see that π_1G is homotopic to a constant. It now follows from our Lemma that $G[K]$ is contained in a single arc component of C.

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REPRESENTATION OF ELLIPTIC OPERATORS IN AN ENVELOPING ALGEBRA.*

By Edward Nelson[1] and W. Forrest Stinespring.[1]

Introduction. In the study of finite-dimensional representations of a Lie group G, the corresponding infinitesimal representation of the Lie algebra and its extension to the enveloping algebra have played an important role. It is reasonable, therefore, to seek aid in the study of infinite-dimensional unitary representations of G from the infinitesimal representation. Bargmann, for example, in his paper [1] finding all irreducible unitary representations of the 2×2 real unimodular group depends heavily on the infinitesimal operators. Unfortunately, the formulation of the infinitesimal representation corresponding to an infinite-dimensional representation is plagued by the fact that in general the image operators are unbounded. Gårding constructed a dense linear manifold of the representation space on which the images of the whole enveloping alegbra are naturally defined, which is invariant under these operators, and on which these operators form a homomorphic image of the enveloping algebra. The theory of Hilbert space, however, leads one to want more. It is desirable for symmetric elements of the enveloping algebra to be represented by self-adjoint operators. The same problem arises in the context of quantum mechanics. In a unitary representation corresponding to some physical particle the images of the elements of the Lie algebra of the underlying symmetry group are momenta of some sort, and certain polynomials in them have physical interpretations. It is appropriate to ask when these operators are observables; that is, when they are essentially self-adjoint, for it is self-adjoint operators that have reasonable spectral resolutions.

Unfortunately, the images of symmetric elements do not always have reasonable interpretations as self-adjoint operators. In fact, Section 5 contains a simple example of such an operator on the Gårding subspace which has no self-adjoint extension at all. I. E. Segal has shown that an operator on the Gårding subspace has a self-adjoint closure provided that it is i times

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* Received July 3, 1958.

[1] National Science Foundation Fellows.

the image of an element of the Lie algebra itself [17], or it is the image of a symmetric element in the center of the enveloping algebra [18]. In this paper we establish the same result for *elliptic* elements of the enveloping algebra and for elements commuting with them. (See Section 2 for precise results.)

Section 3 establishes a connection between second order elliptic elements of the enveloping algebra and the L_1 group algebra. This is applied in Section 4 to construct a class of Lie groups, including the inhomogeneous Lorentz group, which are not CCR groups in the sense of Kaplansky [11]. This section also contains a new proof that semi-simple matrix groups are CCR groups.

1. Notations and terminology. In this paper we shall be concerned with a Lie group G, and elements of the universal enveloping algebra $\mathcal{E}$ of its right-invariant Lie algebra; for brevity, we shall refer to $\mathcal{E}$ as the right-invariant enveloping algebra of G. We denote by $L \to L^+$ the conjugate linear anti-automorphism of $\mathcal{E}$ such that $X^+ = -X$ for X in the Lie algebra. We say that an element of $\mathcal{E}$ is elliptic if it is elliptic as a partial differential operator on G. Let T be a bounded strongly continuous representation of G on a Banach space $\mathcal{V}$. If f is in $L_1(G)$ (an unnamed measure on G will always be left-invariant Haar measure), we write $T(f)$ for the operator $\int_G f(\sigma)T(\sigma)d\sigma$, where this integral converges in the strong operator topology. By the Gårding subspace of $\mathcal{V}$, we mean the linear manifold spanned by all vectors $T(\phi)x$, where x is in $\mathcal{V}$ and ϕ is in the class $C_0^\infty(G)$ of all infinitely differentiable functions on G with compact support. If L is in $\mathcal{E}$, by $dT(L)$ we mean the linear operator with the Gårding subspace for its domain such that $dT(L)(T(\phi)x) = T(L\phi)x$. This is the notation of Segal in [17].

Sometimes it is reasonable to take as the domain of the operator representing L a manifold other than the Gårding subspace. For example:

(1) the well-behaved vectors (see [17]) are a good choice for domain when they are dense;

(2) if T is the regular representation of G on a homogeneous space G/H, then $C_0^\infty(G/H)$ is a natural domain;

(3) if T is the restriction to G of a representation of a larger group, then the Gårding subspace of the larger representation may occur as domain. To handle such cases, we point out the following fact, which we shall use without comment.

THEOREM 1.1. *Let T be a bounded strongly continuous representation of a Lie group G on a Banach space. Let L be an element of the right-invariant enveloping algebra of G. Let A be an operator on the representation space with a single-valued closure such that:*

(1) *A has a dense domain which is invariant under all $T(\phi)$ with ϕ in $C_0^\infty(G)$;*

(2) *$A(T(\phi)x) = T(L\phi)x$ if x is in the domain of A.*

Then the closure of A contains $dT(L)$.

Proof. Let x be any vector in the representation space. Let $x_n \to x$ with x_n in the domain of A. Then for any ϕ in $C_0^\infty(G)$,

$$A(T(\phi)x_n) = T(L\phi)x_n \to T(L\phi)x.$$

2. Elliptic elements of the enveloping algebra and elements commuting with them.

LEMMA 2.1. *Let f be an infinitely differentiable positive definite function on a Lie group G, and let K be an element of the right-invariant enveloping algebra of G. Then $(K^*Kf)(e) \geqq 0$.*

Proof. We approximate K by a right-invariant difference operator. That is to say, for each $h > 0$, we choose numbers $c_\alpha(h)$ for each α in G, so that for fixed h, $c_\alpha(h) \neq 0$ for only finitely many α, and so that for each infinitely differentiable function g,

$$(Kg)(\sigma) = \lim_{h \to 0} \sum_\alpha c_\alpha(h) g(\alpha\sigma)$$

for all σ in G. This can be done so that

$$(K^*g)(\sigma) = \lim_{h \to 0} \sum_\beta \bar{c}_\beta(h) g(\beta^{-1}\sigma)$$

and

$$(K^*Kf)(e) = \lim_{h \to 0} \sum_{\beta\alpha} \bar{c}_\beta(h) c_\alpha(h) f(\alpha\beta^{-1}).$$

By definition of positive definite function, these last sums are non-negative for exery h.

THEOREM 2.2. *Let G be a Lie group, and let U be a strongly continuous unitary representation of G. If L is an elliptic element of the right-invariant enveloping algebra of G, then the closure of $dU(L^*)$ is the adjoint of $dU(L)$.*

Proof. Case I: $L = K^*K$ for some K in the enveloping algebra. We combine two observations to get that $A = dU(L)$ is essentially self-adjoint, i.e. has a self-adjoint closure.

(1) If x is in the Gårding subspace, $(Ax, x) \geqq 0$. For $A = dU(K^*K) = dU(K^*)dU(K)$, so that $(Ax, x) = (dU(K)x, dU(K)x) \geqq 0$ when x is in the Gårding subspace.

(2) $A + 1$ has a dense range. For suppose x is orthogonal to the range of $A + 1$. Then $((A + 1)U(\phi)x, x) = 0$ for all ϕ in $C_0^\infty(G)$. Rewriting yields $\int ((L + 1)\phi)(\sigma)(U(\sigma)x, x)\, d\sigma = 0$; in other words, the function $(U(\cdot)x, x)$ is a weak solution of the partial differential equation $(L + 1)f = 0$. Since $L + 1$ is elliptic, $(U(\cdot)x, x)$ is, in fact, analytic (see, for example, [10]), and $(L + 1)(U(\cdot)x, x) = 0$ in the ordinary pointwise sense. But this contradicts Lemma 1, which shows that $[L(U(\cdot)x, x)](e) \geqq 0$.

These two observations imply that A is essentially self-adjoint, because they show that $(A + 1)^{-1}$ is bounded and densely defined.

Case II: L is a general elliptic element. From Case I we deduce that $dU(L^*L) = dU(L^*)dU(L)$ is essentially self-adjoint. But this fact implies that the closure of $dU(L^*)$ is the adjoint of $dU(L)$ according to the following lemma.

LEMMA 2.3. *Let $\mathfrak{D}$ be a dense linear manifold in a Hilbert space $\mathfrak{H}$. Let T and T' be linear transformations whose domains are $\mathfrak{D}$ and whose ranges are contained in $\mathfrak{D}$ such that T' is contained in the adjoint of T. If $T'T$ is essentially self-adjoint, then the closure of T' is the adjoint of T.*

Proof. We must show that the graph of T^* contains no non-zero element orthogonal to the graph of T'. Suppose $\{a, b\}$ is an element of the graph of T^* that is orthogonal to the graph of T'. In other words, $b = T^*a$, but $(y, a) + (T'y, b) = 0$ for all y in $\mathfrak{D}$. If x is in $\mathfrak{D}$, then Tx is in $\mathfrak{D}$, and hence $(Tx, a) + (T'Tx, b) = 0$; that is, $(x, b) + (T'Tx, b) = 0$. But $1 + T'T$ has a dense range. Consequently $b = 0$. So $(y, a) = 0$ for all y in $\mathfrak{D}$; and, therefore, $a = 0$.

COROLLARY 2.4. *Let G be a Lie group, and let U be a strongly continuous unitary representation of G. Let L be an elliptic element of the right-invariant enveloping algebra $\mathcal{E}$ of G such that $L^* = L$. If M is any element of $\mathcal{E}$ such that $dU(M^*M)$ commutes with $dU(L)$, then the closure of $dU(M^*)$ is the adjoint of $dU(M)$.*

Proof. Let r be an integer greater than the order of M, $A = dU(L^{2r})$, $B = dU(M^*M)$ and $C = A + B$. Then A and C represent elliptic operators, and so are essentially self-adjoint by Theorem 2.2. The operators A and C commute on the Gårding subspace, but we need the stronger fact that their closures $\bar{A}$ and $\bar{C}$ commute, i.e. have commuting spectral resolutions. To see this, notice that the bounded operators $(1+A)^{-1}(1+C)^{-1}$ and $(1+C)^{-1}(1+A)^{-1}$ agree on their common domain, the range of $(1+A)(1+C) = (1+C)(1+A)$. But $(1+A)(1+C)$ represents an elliptic operator, and so is essentially self-adjoint by Theorem 2.2, and thus has a dense range (by the positivity of $A + C + AC$). Consequently $\bar{A}$ and $\bar{C}$ commute.

Let $B_1 = \bar{A} - \bar{C}$. This is essentially self-adjoint, by the spectral theorem. If we show that B_1 is contained in $\bar{B}$, then the corollary will follow, by Lemma 2.3. Let x be in the domain of B_1. Then x is in the domain of $\bar{C}$, so there is a sequence $\{x_n\}$ in the Gårding subspace such that $x_n \to x$ and $Cx_n \to \bar{C}x$. Now for all y in the Gårding subspace, $\| Ay \| \leqq \| Cy \|$, since

$$(Cy, Cy) = (Ay, Ay) + (By, By) + 2(BdU(L^r)y, dU(L^r)y)$$

by the commutativity of $dU(L)$ and $dU(M^*M)$, and all three terms are positive. In particular this is true for $y = x_n - x_m$, so that for the same sequence $\{x_n\}$ we also have $Ax_n \to \bar{A}x$, and so $Bx_n \to B_1x$. That is, B_1 is contained in the closure of B, concluding the proof.

As examples to Corollary 2.4, notice that if G is Abelian or compact then the enveloping algebra $\mathcal{E}$ contains a central symmetric elliptic element, so that for every M in $\mathcal{E}$ the closure of $dU(M^*)$ is the adjoint of $dU(M)$, as is well-known. Also if G is aribtrary but M is central, then M commutes with a symmetric elliptic element, and the same conclusion follows, yielding a result of Segal [18]. If G is semi-simple with maximal essential compact subgroup K, then the Casimir operator on G is of the form $-\Delta_1 + \Delta_2$, where Δ_1 is a central elliptic element of the enveloping algebra of K and $\Delta_1 + \Delta_2$ is elliptic on G. Any M in the enveloping algebra of G which commutes with Δ_1 also commutes with $\Delta_1 + \Delta_2$ since $-\Delta_1 + \Delta_2$ is central, and therefore the closure of $dU(M^*)$ is the adjoint of $dU(M)$.

By Theorem 1.1, all of the above examples persist on the Gårding subspace of a representation of a larger group containing G. In particular, if X is in the Lie algebra of a Lie group and p is any real polynomial, then $dU(p(iX))$ is essentialy self-adjoint since $p(iX)$ is in the enveloping algebra of the one-parameter (Abelian) subgroup generated by X. For iX itself this was established by Segal [17].

3. Second order elliptic operators. In this section some additional information is derived in the second order case using the maximum principle. This will be applied in the following section.

THEOREM 3.1. *Let G be a Lie group, let $X_1, \cdots, X_n$ be a basis for the right-invariant Lie algebra of G, and let $D = X_1^2 + \cdots + X_n^2 + X$, where X is in the Lie algebra. Let T be a bounded strongly continuous representation of G on a Banach space $\mathcal{V}$. Then the closure A of $dT(D)$ is precisely the infinitesimal generator (in the sense of Hille* [9] *and Yosida* [23]) *of a bounded strongly continuous one-parameter semigroup of operators on $\mathcal{V}$. The resolvent $R_\lambda = (\lambda - A)^{-1}$ for $\lambda > 0$ is $T(k_\lambda)$, where k_λ is in $L_1(G)$.*

Proof. First, we shall show that $dT(D) - \lambda$ has a dense range. To do so, we show that any continuous linear functional ξ which annihilates the range must be 0. For if x is in $\mathcal{V}$, the equation

$$0 = \xi(T((D-\lambda)\phi)x) = \int (D-\lambda)\phi(\sigma)\xi(T(\sigma)x)\,d\sigma,$$

which must hold for all ϕ in $C_0^\infty(G)$, says that the function $f(\sigma) = \xi(T(\sigma)x)$ is a weak solution of the elliptic equation $(D^+ - \lambda)f = 0$. By [10], f is analytic, and $(D^+ - \lambda)f = 0$ in the ordinary pointwise sense. But we shall prove that 0 is the only bounded solution of this equation, and in this proof we need consider only real functions. Let U be the open unit ball in a C^∞ coordinate neighborhood of e. Then the following Dirichlet problem has a solution (see [3]): for each continuous function θ on the boundary Γ of U, there is a unique continuous function g on $\bar{U}$ such that g agrees with θ on Γ and satisfies $(D^+ - \lambda)g = 0$ in U. If $\theta \geqq 0$, then $0 \leqq g(e) \leqq \max \theta$, for g can have neither a strictly positive maximum nor a strictly negative minimum inside U, since $D^+g(\sigma) \leqq 0$ at a maximum and $D^+g(\sigma) \geqq 0$ at a minimum. By the Riesz-Markoff theorem, there is a positive measure ν on Γ such that $g(e) = \int_\Gamma \theta\,d\nu$. Now $\nu(\Gamma) < 1$, since $g(e) < 1$ when $\theta = 1$ on all of Γ, again because g cannot have a strictly positive maximum in U. But since the function f considered above satisfies $(D^+ - \lambda)f = 0$ on all of G, then $f(\tau) = \int_\Gamma f(\sigma\tau)\,d\nu(\sigma)$ for all τ in G. Since f is bounded, $\|f\|_\infty \leqq \nu(\Gamma)\|f\|_\infty$, and $f = 0$. So $f(e) = \xi(x) = 0$ for all x in $\mathcal{V}$. Therefore $\xi = 0$, and we have proved that $dT(D) - \lambda$ has a dense range.

Next, we shall conclude the proof of Theorem 3.1 in the special case that $\mathcal{V}$ is $C(G)$, the space of all continuous functions h on G vanishing at ∞, with the supremum norm, and $(T(\sigma)h)(\tau) = h(\sigma^{-1}\tau)$. Again by the maxi-

mum principle, $\|(A-\lambda)\phi\|_\infty \geqq \lambda\|\phi\|_\infty$ for all ϕ in $C_0^\infty(G)$. Since $A-\lambda$ has a dense range, $R_\lambda = (\lambda - A)^{-1}$ is an everywhere-defined operator with $\|R_\lambda\| \leqq 1/\lambda$. By Yosida's theorem [23], this means that A is the infinitesimal generator of a strongly continuous semigroup of contraction operators on $C(G)$. The operator R_λ is a bounded operator on $C(G)$ which commutes with right translations. An application of the Riesz-Markoff theorem shows that R_λ is left convolution by a finite regular measure μ_λ. A simple computation shows that on $G-\{e\}$, μ_λ is a weak solution of the elliptic equation $(\lambda - D)f = 0$. By [10], there is an analytic function k_λ on $G-\{e\}$, such that on $G-\{e\}$, μ_λ is given by $d\mu(\sigma) = k_\lambda(\sigma)\,d\sigma$. Therefore, on the whole of G,

$$d\mu_\lambda(\sigma) = a_\lambda d\delta(\sigma) + k_\lambda(\sigma)\,d\sigma,$$

where a_λ is a number, and δ is the unit mass at e. It is well-known that $a_\lambda = 0$. One way to see this is to consider the operator R of left convolution by μ_λ acting on the Hilbert space $L_2(G)$. If s is an integer $> n/4$ ($n = \dim G$), by [14], R^s has only continuous functions in its range; and then by Theorem 4 of [13], R^s is a Carleman operator. In other words, R^s is left convolution by a function in $L_2(G)$; but it is also left convolution by a measure of the form $d\nu(\sigma) = a_\lambda{}^s d\delta(\sigma) + h_\lambda(\sigma)\,d\sigma$ with h_λ in $L_1(G)$. So $a_\lambda = 0$. For use in the next paragraph we note that $\|k_\lambda\|_1 \leqq 1/\lambda$ since $\|R_\lambda\| \leqq 1/\lambda$.

We return to a general representation T. We shall define R_λ as $T(k_\lambda)$, where k_λ is the function we have just constructed, and we shall prove that $R_\lambda = (\lambda - A)^{-1}$. But for x in $\mathcal{V}$ and ϕ in $C_0^\infty(G)$,

$$R_\lambda(\lambda - A)T(\phi)x = T(k_\lambda{}^*(\lambda - D)\phi)x = T(\phi)x.$$

Hence $R_\lambda = (\lambda - A)^{-1}$, since $\lambda - A$ is closed and has a dense range. Setting $M = \sup_{\sigma \in G}\|T(\sigma)\|$, we have $\|R_\lambda{}^q\| \leqq \|k_\lambda\|_1{}^q M \leqq M\lambda^{-q}$ for all integers $q > 0$. This shows (see [15]) that A is the infinitesimal generator of a bounded strongly continuous semigroup of operators on $\mathcal{V}$.

In the course of the proof of Theorem 3.1 we have established the following corollary which we state briefly for use in Section 4.

COROLLARY 3.2. *In the notation of Theorem* 3.1, *if* s *is an integer* $> n/4$, *then* $R_\lambda{}^s = T(h)$, *where* h *is in both* $L_1(G)$ *and* $L_2(G)$.

4. CCR groups. Kaplansky [11] calls a locally compact group G a *CRR group* if for each function f in $L_1(G)$ and each strongly continuous irreducible unitary representation U of G, the operator $U(f)$ is completely

continuous. The letters CCR, referring to the group algebra $L_1(G)$, stand for completely continuous representation. We have the following criterion for a Lie group to be CCR.

THEOREM 4.1. *Let U be a strongly continuous unitary representation of a Lie group G. Let L be any element of the right-invariant enveloping algebra of G. If $dU(L)$ has an inverse which is contained in a completely continuous operator, then $U(f)$ is completely continuous for all f in $L_1(G)$. Conversely, suppose that $U(f)$ is completely continuous for all f in $L_1(G)$. Then the closure of $dU(\Delta - 1)$ has a completely continuous inverse, where $\Delta = X_1^2 + \cdots + X_n^2$ and $X_1, \cdots, X_n$ is a basis for the right-invariant Lie algebra of G.*

Proof. Let B be the completely continuous operator containing the inverse of $dU(L)$. If ϕ is in $C_0^\infty(G)$, then $U(\phi)$ is completely continuous because $U(\phi) = BU(L\phi)$. But $C_0^\infty(G)$ is dense in $L_1(G)$, and the completely continuous operators are closed in the uniform operator topology. The converse direction follows immediately from Theorem 3.1.

THEOREM 4.2. *Let G be a Lie group, and let K be a compact subgroup of G. Suppose that U is a strongly continuous unitary representation of G whose restriction to K contains each irreducible representation of K only finitely many times. Then $U(f)$ is completely continuous for all f in $L_1(G)$. Suppose, further, that the restriction of U to K is contained in the regular representation of K. Let f be a function on G with compact support. If f has $2s$ continuous derivatives, where s is an integer $> 1/4$ times the dimension of K, then $U(f)$ is of Hilbert-Schmidt type. If f has $4s$ continuous derivatives, then $U(f)$ has an absolutely convergent trace.*

Proof. Let $X_1, \cdots, X_r$ be a basis for the right-invariant Lie algebra of K, and let $\Delta = X_1^2 + \cdots + X_r^2$. By Theorem 3.1, there is a function k in $L_1(K)$ such that $V(k)$ contains the inverse of $dU(\Delta - 1)$, where V is the restriction of U to K. But $V(k)$ must be completely continuous because $L(k)$ is, where L denotes the left regular representation of K, and because each irreducible representation of K is contained in the left regular representation. Consequently, $U(f)$ is completely continuous for all f in $L_1(G)$ by Lemma 4.1.

Now suppose further that V is contained in L. According to Corollary 3.2, if s is an integer $> r/4$, then there is a function h in $L_2(K)$ such that $L(h)$ is the closure of the inverse of $dL((\Delta - 1)^s)$. Since K is compact, $L(h)$ is of Hilbert-Schmidt type; and. a fortiori, $V(h)$ is also. Let f be a

function on G with compact support. If f has $2s$ continuous derivatives, then $U(f) = V(h)U((\Delta - 1)^s f)$, and so $U(f)$ is of Hilbert-Schmidt type. If f has $4s$ continuous derivatives, then $U(f) = [V(h)]^2 U((\Delta - 1)^{2s} f)$, and so $U(f)$ has an absolutely convergent trace.

We remark that in some sense f needs its derivatives only in the direction of K, but we do not make the remark precise.

If G is a connected semi-simple matrix group, then, according to a result of Harish-Chandra [7], the hypotheses of Theorem 4.2 hold for all strongly continuous irreducible unitary representations of G, where K is a maximal compact subgroup. An idea of Godement yields a much shorter proof of this fact [6]; a further simplification of Godement's proof is given in [20]. The conclusion in this case overlaps other results of Harish-Chandra [8]. As another example, Godement [6] has shown that a Lie group of the form $K \cdot A$, where K is compact and A is Abelian, satisfies the hypotheses of Theorem 4.2 for all strongly continuous irreducible representations. The Euclidean group of real n-space, and the inhomogeneous unitary group of complex n-space are examples of such groups.

We may also use Theorem 4.1 to show that certain Lie groups are not CCR groups.

THEOREM 4.3. *Let H be a closed subgroup of the general linear group of a finite-dimensional real vector space $\mathfrak{V}$. Let G be the corresponding inhomogeneous group; i.e. the group of all affine transformations $x \to Ax + b$, where A is in H and b is in $\mathfrak{V}$. Let $\mathfrak{V}^*$ be the dual space of $\mathfrak{V}$, and let $\mathfrak{M}$ be a conical (i.e. invariant under $\xi \to \lambda\xi$ for $\lambda > 0$) submanifold of $\mathfrak{V}^*$ invariant under H^*, the contragredient group of H. Let μ be a regular measure on $\mathfrak{M}$ whose null sets are invariant under H^* and also under the maps $x \to \lambda x$ for $\lambda > 0$. For σ in G, say $\sigma x = Ax + b$, let $U(\sigma)$ be the unitary operator on $L_2(\mathfrak{M}, \mu)$ mapping*

$$h(\xi) \to h(A^*\xi)[\exp(i\xi(b))][d\mu(A^*\xi)/d\mu(\xi)]^{\frac{1}{2}},$$

where A^* is the adjoint transformation to A. Then representation U has the property that there exists a function f in $L_1(G)$ such that $U(f)$ is not completely continuous.

Proof. Let Δ_H be the sum of the squares of the elements of a basis for the Lie algebra of H. Choose a positive definite inner product in $\mathfrak{V}$, and let Δ be the element of the enveloping algebra of the translation part of G corresponding to the Laplacian operator on $\mathfrak{V}$. Let N be the closure of

$dU(\Delta_H - 1)$, and M, the closure of $dU(\Delta)$. Then M is multiplication by the function $-r^2$ on $\mathcal{V}^*$, where $r(\xi) = |\xi|$. Any one-to-one measurable transformation S of $\mathcal{M}$ onto itself which preserves null sets of μ gives rise to a unitary operator $\hat{S}$ on $L_2(\mathcal{M}, \mu)$ as follows:

$$(\hat{S}h)(\xi) = h(S\xi)[d\mu(S\xi)/d\mu(\xi)]^{\frac{1}{2}}.$$

If S_1 and S_2 commute, so do $\hat{S}_1$ and $\hat{S}_2$. For $\lambda > 0$, let T_λ be the transformation of $\mathcal{M}$ given by $T_\lambda(\xi) = \lambda\xi$. By the remark we have just made, $\hat{T}_\lambda$ commutes with $U(H)$, and therefore with N. Let ϕ be in $C_0^\infty(\mathcal{M})$ with $\|\phi\|_2 = 1$. Then $\|N\hat{T}_\lambda\phi\|_2 = \|\hat{T}_\lambda N\phi\|_2 = \|N\phi\|_2$, and $\|M\hat{T}_\lambda\phi\|_2 \leqq \rho(\lambda)\|\hat{T}_\lambda\phi\|_2 = \rho(\lambda)$, where $\rho(\lambda)$ is the maximum of $|\xi|^2$ when $\lambda\xi$ is in the support of ϕ. Consequently, $(N + M)\hat{T}_\lambda\phi$ is bounded for $\lambda \geqq 1$. But the $\hat{T}_\lambda\phi$ for $\lambda \geqq 1$ are not contained in any compact set; in fact, a suitable subfamily forms an infinite orthonormal set. In other words, the inverse of $N + M$ is not contained in any completely continuous operator. Theorem 3.1 asserts that this fact implies the conclusion of the present theorem.

If the group H^* is transitive on the cone $\mathcal{M}$, then the representation U of G is irreducible (see [12]), and hence G is not CCR. As examples of linear groups H such that H^* is transitive on a cone, we mention the (real or complex) general linear group, the (real or complex) unimodular group acting on a space of dimension $\geqq 2$, a non-compact pseudo-orthogonal group, and the group of transformations $x \to ax$ for $a > 0$ acting on the line. This last case shows that the group of transformations $x \to ax + b$ for $a > 0$ acting on the line is *not* CCR despite the contrary assertion by Gelfand and Neumark in the introduction to [4]. The next to last case shows that the inhomogeneous Lorentz group is not CCR. Theorem 4.3 says that the irreducible representation of this group over the light-cone with spin 0 (see [2] and [22]) does not universally send integrable functions on the group into completely continuous operators. The method of proof of Theorem 4.3 shows that the same is also true for irreducible representations of higher spin over the light-cone.

5. Counterexamples. A unitary representation of a Lie group does not always send symmetric elements of the enveloping algebra into operators essentially self-adjoint on the Gårding subspace. The first such counterexample was given by von Neumann (unpublished). Let G be the group of the Heisenberg commutation relations; that is to say, G is the 3-dimensional group of all real matrices of the form:

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

This group G has an irreducible unitary representation on $L_2(-\infty, +\infty)$ whose infinitesimal operators are d/dx, ix, and i. In other words, the enveloping algebra of G maps onto all ordinary differential operators with polynomial coefficients, and there are many of these which are symmetric but not essentially self-adjoint.

In [17], Segal proved that the elements of the Lie algebra itself multiplied by i are always represented by operators essentially self-adjoint on the Gårding subspace. He also asserted in [17] that symmetric second order elements of the enveloping algebra are always represented by operators essentially self-adjoint on the Gårding subspace if the group is unimodular. However, L. Gross found an error in the proof, and this observation formed the starting point for our researches. (The statement on line 5 of p. 236 of [17] is incorrect, since the quantity on the preceding line is only $0(n)$.) In fact, we shall present a counterexample in the next paragraph even for a semi-simple group. The remainder of [17] is unaffected, since one may use instead Segal's theorem that central elements of the enveloping algebra are always represented by essentially self-adjoint operators if they are symmetric [18] (see a forthcoming note by Segal [19]).

Let G be the group of all 2×2 real unimodular matrices. Bargmann [1] has found all the irreducible unitary representations of this group. The one he denotes by $C^0{}_{1/4}$ acts on $L_2(M)$, where M is the real numbers modulo 2π, and is given infinitesimally by

$$dU(\Lambda_0) = -d/dx$$

$$dU(\Lambda_1) = \cos x\, d/dx - \tfrac{1}{2}\sin x$$

$$dU(\Lambda_2) = \sin x\, d/dx + \tfrac{1}{2}\cos x,$$

where we denote the representation by U. Let $A = \Lambda_0\Lambda_2 + \Lambda_2\Lambda_0$. Calculation shows

$$dU(A) = -2d/dx \sin x\, d/dx + \tfrac{1}{2}\sin x.$$

Let E be the operator with the same formula as $dU(A)$ but with domain $C_0^\infty(M)$; E and $dU(A)$ have the same closure. The operator E is not essentially self-adjoint because of the vanishing of $\sin x$. One way to see this is to consider the functions

$$f(x) = \begin{cases} \log|\sin x| & \text{for } -\tfrac{\pi}{2} \leqq x \leqq \tfrac{\pi}{2} \\ 0 & \text{elsewhere on } M, \end{cases}$$

and

$$g(x) = \begin{cases} 1 & \text{for } 0 \leqq x \leqq \pi \\ 0 & \text{elsewhere on } M. \end{cases}$$

Then the distribution derivative (see [16]) of f is the map

$$\phi \to \text{P. V.} \int_{-\pi/2}^{\pi/2} \phi(x) \cos x / \sin x \, dx,$$

where P. V. means the Cauchy principal value of the integral. Hence the operator with the same formula as E applied to f in the sense of distributions yeilds the function

$$h(x) = \begin{cases} 2\sin x + \tfrac{1}{2}\sin x f(x) & \text{for } -\tfrac{\pi}{2} \leqq x \leqq \tfrac{\pi}{2} \\ 0 & \text{elsewhere on } M. \end{cases}$$

Since f and h are both in $L_2(M)$, this means that $h = E^*f$. Similarly, $k = E^*g$. where

$$k(x) = \begin{cases} \tfrac{1}{2}\sin x & \text{for } 0 \leqq x \leqq \pi \\ 0 & \text{elsewhere on } M. \end{cases}$$

But $(E^*f, g) - (f, E^*g) = \int_0^{\pi/2} 2 \sin x \, dx \neq 0$. Consequently E, and hence $dU(A)$, are not essentially self-adjoint.

To see how ellipticity prevents this sort of thing from happening, let B be an elliptic element in the enveloping algebra of G. By definition, $B = p(\Lambda_0, \Lambda_1, \Lambda_2) +$ lower order terms, where p is a homogeneous polynomial that never vanishes when its argument is a non-zero triple of real numbers. Then

$$dU(B) = p(-d/dx, \cos x \, d/dx, \sin x \, d/dx) + \text{lower order terms}$$

is an ordinary differential operator on M with a nowhere vanishing leading coefficient. In this case the conclusion of Theorem 2.2, that the closure of $dU(B)$ is the Hilbert space adjoint of its formal Lagrange adjoint, is well-known.

For another example of a symmetric second-order element of an enveloping algebra being represented by an operator which is not essentialy self-adjoint, let G be the $ax + b$ group. Let U be one of its two infinite dimensional irreducible unitary representations (see [12]). One version of U acts on $L_2(-\infty, +\infty)$ and has infinitesimal operators

$$dU(X) = d/dx \text{ and } dU(Y) = ie^x.$$

Then $dU(XY + YX) = 2ie^x\, d/dx + ie^x$, and by [21] (X, sec. 2) this operator has deficiency indices $(0, 1)$, since e^{-x} is not integrable near $-\infty$ but is integrable near $+\infty$. Consequently, $dU(XY + YX)$ has no self-adjoint extension. Although G is not unimodular, this example can be converted into one for a unimodular group. A theorem of Gleason [5] shows that there is a unimodular group G_1 with a normal subgroup R isomorphic to the additive group of all real numbers such that $G \cong G_1/R$. Then U composed with the natural map $G_1 \to G$ gives a representation of G_1 with the same infinitesimal operators.

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KRONECKERIAN MODEL OF FIELDS OF ELLIPTIC MODULAR FUNCTIONS.*[1]

By JUN-ICHI IGUSA.

In our previous paper [7-III], we gave a geometric construction of the field of elliptic modular functions of an arbitrary level n for any characteristic which does not divide n and we discussed its basic properties. We shall now piece together the fields of modular functions of level n in different characteristics so that we get a fibre system parametrized by the ring of rational integers. More precisely, we shall construct a non-singular projective model of the field of modular functions of level n in characteristic zero over the field of rational numbers such that its reduction with respect to every prime number p not dividing n is a non-singular projective model of the field of modular functions of level n in characteristic p over the prime field. We can also say, more suggestively, that *the compactified fundamental domain of the congruence modular group of level n admits a "characteristic p deformation," provided p does not divide n.* At any rate, the construction is unique up to a biregular transformation. The birational geometry of this type of fibre systems explains, for instance, why the absolute invariants of three elliptic curves obtained from the invariant transformation equations of degree 11, 17, 19 are rational numbers with denominators exactly at these prime numbers. Moreover, the existence of the fibre system guarantees that the Petersson conjecture [10] on the eigen values of Hecke operators operating on cusp forms is true for dimension -2. As we know, the validity of this conjecture for *almost all* prime numbers was proved by Eichler [4] in a slightly less general case, which was later generalized by Shimura [12]. We like to mention, however, that the validity for *individual* prime numbers was an open question.

1. Preparations. Let Z be the ring of rational integers and let Q be its field of fractions. We shall denote the algebraic closure of Q by Q_0. Also,

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we fix a positive integer n once for all and we shall denote the field of n-th roots of unity over Q by Q_n. Let j be transcendental over Q and pick an elliptic curve A_j which is defined over $Q(j)$ and which has j as its absolute invariant. We note that A_j is unique only up to an isomorphism. Let Ω be the group of points of A_j of order n. We shall denote by Ku the identification mapping of two points of opposite signs of an elliptic curve. Then $Q_0(j, Ku(\Omega))$ is what we called the field of modular functions of level n in characteristic zero.[2] In fact, if we extend Q_0 to the field of complex numbers, we get the classical field of modular functions of level n. We know that the field $Q(j, Ku(\Omega))$ does not depend on the choice of A_j and that Q_n is the algebraic closure of Q in $Q(j, Ku(\Omega))$.

Now, pick two generators ω_1 and ω_2 of Ω and let j_n be the absolute invariant of the factor group of A_j by the cyclic group generated by ω_1. We shall show that the field $K = Q(j, j_n, Ku(\omega_2))$ depends neither on the choice of A_j nor on the choice of the generators ω_1 and ω_2 of Ω. Suppose first that ω_1' and ω_2' are two generators of Ω. Then we can transform the pair (ω_1, ω_2) either to (ω_1', ω_2') or to $(-\omega_1', -\omega_2')$ by an automorphism of $Q(j, \Omega)$ over $Q(j)$ [7-III, Theorem 3]. If j_n' is the absolute invariant of the factor group of A_j by the cyclic group generated by ω_1', this automorphism transforms $(j_n, Ku(\omega_2))$ to $(j_n', Ku(\omega_2'))$. Therefore, if we fix A_j, the field K is unique up to an isomorphism over $Q(j)$. Next, let B_j be another elliptic curve which is defined over $Q(j)$ and which has j as its absolute invariant. Let ω_1' and ω_2' be the images of ω_1 and ω_2 under an isomorphism between A_j and B_j. Then ω_1' and ω_2' generate the group of points of B_j of order n. Let j_n' be the absolute invariant of the factor group of B_j by the cyclic group generated by ω_1'. Then the conjugate of ω_1 over $Q(j, j_n)$ is simply another generator of the cyclic group generated by ω_1 [7-III, remark after Theorem 7]. The same is true for ω_1' instead of ω_1 over $Q(j, j_n)$, hence j_n' is contained in $Q(j, j_n)$. Since $Ku(\omega_2')$ is also rational over $Q(j, Ku(\omega_2))$, we see that $Q(j, j_n', Ku(\omega_2'))$ is contained in $Q(j, j_n, Ku(\omega_2)) = K$. Since the situation is symmetric, we get $Q(j, j_n', Ku(\omega_2')) = K$. This proves the assertion.

We have thus shown that *the field K is an intrinsically defined algebraic extension of $Q(j)$*. If we apply the Galois theory [7-III, Theorem 3], we see that $Q(j, Ku(\Omega))$ and $Q(j)$ are respectively the compositum and the inter-

[2] The properties of fields of modular functions we need in this paper are discussed purely geometrically in [7-III]. If the reader does not mind using transcendental theory in the case of characteristic zero, he can refer to the classical treatment like [14]. However, at least at present, the theory of modular functions in positive characteristic is indispensable to prove our main theorem.

section of K and $Q_n(j)$. Therefore K and $Q_n(j)$ are linearly disjoint over $Q(j)$, hence K and Q_n are linearly disjoint over Q. In particular, K is a regular extension of Q.

Let p be a prime number and extend the p-adic valuation of Q to $Q(j)$ by the Gaussian definition. Let $Q(j)_p$ be the corresponding valuation ring of $Q(j)$. If F is the prime field of characteristic p, i.e., if we put $F = Z/pZ$, and if j' is transcendental over F, then $Q(j)_p$ is the specialization ring of the specialization $Z[j] \to F[j']$. This being remarked, we shall prove the following lemma:

LEMMA 1. *If p does not divide the level n, the integral closure R of $Q(j)_p$ in K is an unramified discrete valuation ring.*

Proof. Suppose first that p is different from 2. Let λ be a root of the equation

$$2^8(T^2-T+1)^3-jT^2(T-1)^2=0$$

and let R_λ be an extension of $Q(j)_p$ to the field $K(\lambda)$. It is then sufficient to show that this valuation ring is unique and unramified. It is equivalent to say that the residue field of R_λ is of degree $[K(\lambda):Q(j)]$ over $F(j')$. In fact, we have only to recall the well-known relation between total degree and local degrees composed of residue degrees and ramification indices [cf. 18]. Let λ' be the specialization of λ over the residue homomorphism of R_λ. Since 2 is a unit in $Q(j)_p$, hence also in R_λ, we see that λ' is transcendental over F and we have $[F(\lambda'):F(j')]=[Q(\lambda):Q(j)]$. Therefore, we have only to show that the residue field of R_λ is of degree $[K(\lambda):Q(\lambda)]$ over $F(\lambda')$. Consider a plane cubic A_λ which is defined over $Q(\lambda)$ by the equation $Y^2Z=X(X-Z)(X-\lambda Z)$. We introduce a group law in A_λ, say, with reference to the point $(0,1,0)$. We know that A_j and A_λ are isomorphic over some quadratic extension of $Q(\lambda)$. Let ϖ_1 and ϖ_2 be the images of ω_1 and ω_2 under an isomorphism between A_j and A_λ. Then ϖ_1 and ϖ_2 generate the group of points of A_λ of order n. Moreover, we have $Q(\lambda, Ku(\varpi_2))=Q(\lambda, Ku(\omega_2))$, hence $Q(\lambda, j_n, Ku(\varpi_2))=Q(\lambda, j_n, Ku(\omega_2))=K(\lambda)$. Here j_n can be considered as the absolute invariant of the factor group of A_λ by the cyclic group generated by ϖ_1. Let $(\lambda', j_n', Ku(\varpi_2)')$ be the specialization of $(\lambda, j_n, Ku(\varpi_2))$ over the residue homomorphism of R_λ. Then A_λ is reduced to a plane cubic $A_{\lambda'}$ which is defined over $F(\lambda')$ by the equation $Y^2Z=X(X-Z)(X-\lambda' Z)$. We introduce a group law in $A_{\lambda'}$ with reference to the point $(0,1,0)$. Let (ϖ_1', ϖ_2') be a specialization of (ϖ_1, ϖ_2) over the above specialization. Then ϖ_1' and ϖ_2' generate the group of points of $A_{\lambda'}$ of order n. Moreover, j_n' is the absolute invariant of the factor group of $A_{\lambda'}$ by the cyclic group

generated by ϖ_1', while we have $Ku(\varpi_2)' = Ku(\varpi_2')$. On the other hand, from the Galois theory [7-III, Theorem 1] we conclude that

$$[F(\lambda', j_n', Ku(\varpi_2')) : F(\lambda')] \text{ and } [Q(\lambda, j_n, Ku(\varpi_2)) : Q(\lambda)]$$

are equal. Therefore $F(\lambda', j_n', Ku(\varpi_2'))$ must coincide with the residue field of R_λ and it is of degree $[K(\lambda) : Q(\lambda)]$ over $F(\lambda')$. This completes the proof in the case p is different from 2. Suppose next that p is 2. Let μ be a root of the equation

$$3^3T^3(T^3+2^3)^3 - j(T^3-1)^3 = 0$$

and let R_μ be an extension of $Q(j)_p$ to $K(\mu)$. Let μ' be the specialization of μ over the residue homomorphism of R_μ. Then, by a similar reason as in the previous case, it is sufficient to show that the residue field of R_μ is of degree $[K(\mu) : Q(\mu)]$ over $F(\mu')$. In order to prove this fact, we consider a plane cubic A_μ which is defined over $Q(\mu)$ by the equation $X^3 + Y^3 + Z^3 = 3\mu XYZ$. We introduce a group law in A_μ, say, with reference to the point $(-1, 1, 0)$. We know that A_j and A_μ are isomorphic over some quadratic extension of $Q(\mu)$. Let ϖ_1 and ϖ_2 be the images of ω_1 and ω_2 under an isomorphism between A_j and A_μ. Then we have $Q(\mu, j_n, Ku(\varpi_2)) = K(\mu)$. Let $(\mu', j_n', Ku(\varpi_2)')$ be the specialization of $(\mu, j_n, Ku(\varpi_2))$ over the residue homomorphism of R_μ. Let (ϖ_1', ϖ_2') be a specialization of (ϖ_1, ϖ_2) over this specialization. Then, from the Galois theory [7-III, Theorem 2] we conclude that $F(\mu', j_n', Ku(\varpi_2'))$ is the residue field of R_μ and it is of degree $[K(\mu) : Q(\mu)]$ over $F(\mu')$.

Now, consider the ring $Z[j]$ of polynomials in j with integral rational coefficients and take its integral closure S in K. Then S is a $Z[j]$-module and a general theorem guarantees the finiteness of S over $Z[j]$. In this case, however, the proof is quite simple. Pick a base $\theta_1, \theta_2, \cdots$ of K over $Q(j)$ from S and let d be the discriminant of the base. Then d is different from zero and S is contained in the finite $Z[j]$-module generated by $\theta_1/d, \theta_2/d, \cdots$. Since $Z[j]$ is Noetherian, we see that S itself is a finite $Z[j]$-module. If we treat j and $1/j$ symmetrically, we are forced to introduce the integral closure N of $Z[1/j]$ in K. Then N is a finite $Z[1/j]$-module. On the other hand, since $Z[j]$ and $Z[1/j]$ are contained in $Q(j)_p$, both S and N are contained in the integral closure R of $Q(j)_p$ in K. If p does not divide n, we know by Lemma 1 that R is an unramified discrete valuation ring, i.e. that the ideal of non-units of R is pR. We shall show that we have $pR \cap S = pS$. Let f be an element of $pR \cap S$. Then f/p is contained in R. Since R is the only discrete valuation ring of K which comes from a minimal prime ideal of pS, we see that f/p is contained in every discrete valuation ring of K which comes

from a minimal prime ideal of S. However, since S is an integrally closed Noetherian domain, the intersection of these valuation rings coincides with S by a theorem of Krull [cf. 18]. Therefore f/p is an element of S, hence f is an element of pS. The converse is obvious. Similarly, we get $pR \cap N = pN$. In particular, pS and pN are prime ideals of S and N. Let K' be the residue field of R, i.e., put $K' = R/pR$. If S' and N' are the images of S and N under the residue homomorphism $R \to K'$, i.e., if we put $S' = S/pS$ and $N' = N/pN$, then K' is the common field of fractions of S' and N'. Now, we shall prove the following lemma:

LEMMA 2. *The field $K' = R/pR$ is a regular extension of F. Moreover, the compositum of K' and the algebraic closure k of F is the field of modular functions of level n in characteristic p.*

Proof. We shall use the same notation as in the proof of Lemma 1. First we shall show that K' is regular over F. It is sufficient to show that the residue fields R_λ/pR_λ and R_μ/pR_μ are both regular over F. Let F_n be the field of n-th roots of unity over F. If we apply the Galois theory [7-III, Theorem 1], we see that $R_\lambda/pR_\lambda = F(\lambda', j_n', Ku(\varpi_2'))$ and $F_n(\lambda')$ are linearly disjoint over $F(\lambda')$. We know that their compositum is regular over F_n, hence R_λ/pR_λ is regular over F. In the same way, we see that R_μ/pR_μ is regular over F. Next we shall show that the compositum kK' is the field of modular functions of level n in characteristic p. Pick an elliptic curve $A_{j'}$ which is defined over $F(j')$ and which has j' as its absolute invariant. Let Ω' be the group of points of $A_{j'}$ of order n. Suppose that p is different from 2. We know that $A_{j'}$ and $A_{\lambda'}$ are isomorphic over some quadratic extension of $F(\lambda')$. Let ω_1' and ω_2' be the images of ϖ_1' and ϖ_2' under an isomorphism between $A_{j'}$ and $A_{\lambda'}$. Then ω_1' and ω_2' generate the group Ω' and j_n' can be considered as the absolute invariant of the factor group of $A_{j'}$ by the cyclic group generated by ω_1'. Moreover, by the Galois theory [7-III, Theorem 3] we have $k(j', j_n', Ku(\omega_2')) = k(j', Ku(\Omega'))$, and this is the field of modular functions of level n in characteristic p [cf. 7-III]. Therefore, if we put $K'' = F(j', j_n', Ku(\omega_2'))$, we have only to show $kK' = kK''$. Since we have $F(\lambda', Ku(\omega_2')) = F(\lambda', Ku(\varpi_2'))$, we get $F(\lambda', j_n', Ku(\omega_2')) = F(\lambda', j_n', Ku(\varpi_2'))$, hence $K''(\lambda') = R_\lambda/pR_\lambda = K'(\lambda')$. Now, in case n is even, we have $K'(\lambda') = K'$ and $K''(\lambda') = K''$, hence $K' = K''$. In case n is odd, the situation is rather delicate. Consider the composites F_nK'' and $F_nK''(\lambda')$ and put $\mathfrak{Z} = F_nK''$. Then $\mathfrak{Z}$ and $F_n(\lambda')$ are linearly disjoint over $F_n(j')$ with $F_nK''(\lambda')$ as their compositum and the ramification index of $\mathfrak{Z}$ over $F_n(j')$ at $j' = \infty$ is n [7-III, Theorem 6]. We shall show that $\mathfrak{Z}$ is characterized by these properties.

Since $F_n(\lambda')$ is ramified over $F_n(j')$ at $j'=\infty$ with index 2 and since n is odd, we see that the inertia groups of $F_nK''(\lambda')$ over $F_n(j')$ at $j'=\infty$ are cyclic of order $2n$. The elements of order 2 in these groups generate a unique normal subgroup of the Galois group of $F_nK''(\lambda')$ over $F_n(j')$ and Σ is precisely the corresponding invariant subfield of $F_nK''(\lambda')$. Hence Σ is unique. We shall show that F_nK' has the same properties as Σ. We have seen in the proof of Lemma 1 that F_nK' has the first property. On the other hand, F_nK' and Σ both contain the Galois extension of $F_n(j')$ generated by j'_n. This implies that the ramifications of F_nK' and Σ over $F_n(j')$ are the same except possibly at $j'=\infty$ [cf. 7-III, remark after Theorem 7]. If the ramifications are different, the ramification index of F_nK' over $F_n(j')$ at $j'=\infty$ will be $2n$. Then the genus of K' will be larger than the genus of Σ, which is equal to that of K [7-III, Theorem 5]. This is a contradiction [cf. 1]. Hence we get $F_nK'=\Sigma$. Suppose finally that p is 2. Define the generators ω_1' and ω_2' of Ω' and the field K'' as in the previous case. Then the compositum kK'' is the field of modular functions of level n in characteristic 2, and we get $K''(\mu')=R_\mu/pR_\mu=K'(\mu')$. Now, in case n is a multiple of 3, we have $F_3K'(\mu')=F_3K'$ and $F_3K''(\mu')=F_3K''$, hence $F_3K'=F_3K''$. If n is not a multiple of 3, then $F_{3n}K''$ and $F_{3n}(\mu')$ are linearly disjoint over $F_{3n}(j')$ with $F_{3n}K''(\mu')$ as their compositum and the ramification index of $F_{3n}K''$ over $F_{3n}(j')$ at $j'=\infty$ is n. As before, these properties characterize $F_{3n}K''$ and the first property is satisfied by $F_{3n}K'$. On the other hand, $F_{3n}K'$ and $F_{3n}K''$ both contain the Galois extension of $F_{3n}(j')$ generated by j_n' from which we conclude that the second property is also satisfied by $F_{3n}K'$. Hence we get $F_{3n}K'=F_{3n}K''$.

We note that the main complications of the proofs of the two lemmas result from the inclusion of the two cases $p=2,3$. If the reader does not mind excluding these cases, we can give very simple proofs.

2. Kroneckerian model. Let K be, as in the previous section, the intrinsically defined algebraic extension of $Q(j)$ depending on n. Also, let S and N be the integral closures of $Z[j]$ and $Z[1/j]$ in K. The *Kroneckerian model of fields of modular functions* of *level* n is, then, the union $\mathfrak{M}$ of specialization rings of S and N. We note that the union $\mathfrak{D}$ of specialization rings of $Z[j]$ and $Z[1/j]$ is the so-called universal projective straight line and the Kroneckerian model $\mathfrak{M}$ is the "derived normal model" of $\mathfrak{D}$ in K. At any rate, if we classify specialization rings of $\mathfrak{M}$ according to the characteristics of image fields, the model $\mathfrak{M}$ splits into enumerably many "local models" as follows:

$$\mathfrak{M} = \mathfrak{M}_0 + \sum_p \mathfrak{M}_p.$$

Now, following Zariski [17, see also 9], we shall construct a projective model of $\mathfrak{M}_0$ such that $\mathfrak{M}_p$ will become the union of specialization rings of points of its reduction with respect to p [cf. 11], and we shall investigate the local models closely.

Let m be a positive integer and let S_m be the set of elements f of S such that f/j^m is integral over $Q[1/j]$. Then, the sequence $S_1, S_2, \cdots$ determines an increasing filtration of the Z-module S. Also, it is clear that we have $pS \cap S_m = pS_m$. Let f be an element of S and let $f^t + c_1(j)f^{t-1} + \cdots + c_t(j) = 0$ be an irreducible monic equation for f over $Z[j]$. Then f/j^m is integral over $Q[1/j]$ if and only if we have $\deg. c_i(j) \leqq mi$ for $i = 1, 2, \cdots, t$. This implies that f/j^m is integral over $Z[1/j]$, hence we have $S_m = S \cap j^m N$. Put $N_m = N \cap (1/j)^m S$. Then, the sequence $N_1, N_2, \cdots$ determines an increasing filtration of the Z-module N and we have $pN \cap N_m = pN_m$. Moreover, the multiplication by j^m is an isomorphism of N_m to S_m. Now, assume that p does not divide the level n and let S'' and N'' be the integral closures of $F[j']$ and $F[1/j']$ in K'. Here F, j' and K' are defined in the previous section. We can consider S'' and N'' as the integral closures of $S' = S/pS$ and $N' = N/pN$ in K'. Similarly as above, put $S''_m = S'' \cap (j')^m N''$ and $N''_m = N'' \cap (1/j')^m S''$. Then, we get increasing filtrations of the F-modules S'' and N'' and the multiplication by $(j')^m$ is an isomorphism of N''_m to S''_m. On the other hand, let S'_m and N'_m be the images of S_m and N_m under the residue homomorphism $R \to R/pR = K'$. Here R is defined in Lemma 1. Then, we get increasing filtrations of the F-modules S' and N'. Also, it is clear that S'_m is contained in S''_m and N'_m is contained in N''_m. Since we have $S'_m = S_m/pS_m$, the rank of S'_m as an F-module is equal to the rank of S_m as free Z-module. Similarly, the rank of N'_m is equal to the rank of N_m. Since S_m and N_m are isomorphic, these ranks are the same. Now, from Lemma 2 and from our previous result [7-III, Theorem 5] we conclude that K and K' have the same genus. Therefore, from the Riemann-Roch theorem [cf. 15] we conclude that S_m and S''_m have the same rank at least when $m[K: Q(j)]$ is greater than the degree of the canonical divisor of K. Actually, this condition is always satisfied [7-III, Theorem 4, Theorem 5]. Therefore, we get $S'_m = S''_m$ and $N'_m = N''_m$ for all m, hence $S' = S''$ and $N' = N''$. In other words, *the integral domains S' and N' are integrally closed in K'.* This is a key point.

Now, it is clear that S_m contains $1, j, \cdots, j^m$ and N_m contains $1, 1/j, \cdots, (1/j)^m$. Also, we know that S is a finite $Z[j]$-module and N is a finite

$Z[1/j]$-module. Therefore, if we take m sufficiently large, we have $Z[S_m] = S$ and $Z[N_m] = N$, hence applying the residue homomorphism $R \to K'$, we get $F[S'_m] = S'$ and $F[N'_m] = N'$. This being remarked, let $f_0, f_1, \cdots, f_r$ be a base of the free Z-module S_m. Let C be a curve in the projective space over Q of dimension r with $M = (f_0, f_1, \cdots, f_r)$ as a generic point over Q. It is clear that, once m is fixed, this curve C is uniquely determined up to a projective transformation with integral unimodular coefficient-matrix. We shall show that we can normalize C to the extent that we have $f_0 = 1, f_1 = j, \cdots, f_m = j^m$. We have only to show that the free Z-module generated by $1, j, \cdots, j^m$ is a direct summand of S_m. If it is not so, we can find a prime number p, which might divide n, such that the images of $1, j, \cdots, j^m$ under the residue homomorphism $S_m \to S_m/pS_m$ are no more independent over F. On the other hand, since $Z[j]$ is integrally closed in $Q(j)$, we obviously have $pS \cap Z[j] = pZ[j]$. Therefore, the image of j is transcendental over F, and thus we get a contradiction.

We shall now translate some ring-theoretic properties of S and N into geometric properties of C. We shall first show that C is non-singular. If we take the first co-ordinate hyperplane as a plane at infinity, the corresponding affine co-ordinate ring is simply the tensor product $Q \otimes S$, which is the integral closure of $Q[j]$ in K. Therefore, this affine representative is normal over Q, hence non-singular. If we take the $(m+1)$-th co-ordinate hyperplane as a plane at infinity, the corresponding affine co-ordinate ring is simply the tensor product $Q \otimes N$, and this is the integral closure of $Q[1/j]$ in K. Therefore, this affine representative is also normal over Q, hence non-singular. We note that these open sets already cover the entire curve C. Hence C is non-singular. In the following we shall fix this open covering of C.

We shall next show that, as long as p does not divide n, the reduction C_p of C with respect to p remains non-singular. We know that in such a case pS is a prime ideal of S and that the field of fractions K' of $S' = S/pS$ is regular over F. Therefore, the reduction of one open set of C with respect to p is an irreducible affine curve defined over F. Moreover, since we have $F[S'_m] = S'$ and since S' is integrally closed in K', this curve is normal over F, hence non-singular. In a similar way, we see that the reduction of other open set of C with respect to p is non-singular. These two affine curves determine an open covering of C_p, hence C_p is non-singular. We have thus proved the following theorem the first part of which is clear from the construction:

THEOREM 1. *There exists a non-singular projective model C of the*

field K over Q such that the local model $\mathfrak{M}_p$ is the union of specialization rings of points of its reduction C_p with respect to p. If p does not divide the level n, then C_p is a non-singular projective model of the field K' over F.

We note that in proving this theorem we made a full use of the theory of modular functions in arbitrary characteristic. Conversely, many of the theorems in characteristic p are contained in this theorem. We regard this theorem as one of the fundamental theorems in the theory of elliptic modular functions.

3. A digression. In the previous section, we defined the Kroneckerian model $\mathfrak{M}$ of fields of modular functions of level n. The union $\mathfrak{M}_n$ of local models $\mathfrak{M}_p$ for p dividing n may be called the *frontier* of $\mathfrak{M}$, because *the reduction of C with respect to such p is reducible.* In fact, the field K contains the field $Q(j, j_n)$. Here j_n is the absolute invariant of the factor group of A_j by its cyclic subgroup of order n. If, therefore, we can show that the extension of $pZ[j]$ to the integral closure of $Z[j]$ in $Q(j, j_n)$ is not unique, then pS will be divisible by more than one minimal prime ideal of S and, consequently, the reduction of C with respect to p will contain the same number of components. In order to show that the extension is not unique, we recall that j_n and j are related by an equation $\Phi_n(X, Y) = 0$ over Z known as the *invariant transformation equation* of *degree* n. Moreover [cf. 3], if p^e is the p-primary part of n and if we put $n = n'p^e$, then $\Phi_n(X, Y)$ splits modulo p as follows:

$$\Phi_n(X, Y) \equiv \Phi(X^{p^e}, Y)\Phi(X, Y^{p^e}) \cdot \prod_{i=1}^{e-1} \Phi(X^{p^{e-i-1}}, Y^{p^{i-1}})^{p-1} \mod p.$$

Here $\Phi(X, Y) = 0$ is the invariant transformation equation of degree n', which remains absolutely irreducible modulo p [7-III, remark after Theorem 7]. Therefore, the complete set of conjugates of j_n over $Q(j)$ splits modulo p into $e+1$ complete sets of conjugates over $F(j')$ among which $e-1$ sets appear $(p-1)$-times. This is more than what we actually need here.

Now, since the equation $\Phi_n(X, Y) = 0$ is absolutely irreducible in characteristic zero, it defines a curve Φ_n over Q in the product $D \times D$ of two projective straight lines. In the above discussion, we used a *biregular property* of Φ_n or, more precisely, its locus variety in the product $\mathfrak{D} \times \mathfrak{D}$. We shall now discuss a *birational property* of Φ_n in case n is a prime number.

THEOREM 2. *Let Γ be an arbitrary projective curve which is birationally equivalent to the curve Φ_p over some extension of Q. Then, unless Γ is of*

genus zero, its reduction modulo p is either reducible or an irreducible curve of a smaller genus.

Proof. If we put $n' = e = 1$ in the congruence relation we mentioned before, we get $\Phi_p(X, Y) \equiv (X^p - Y)(X - Y^p) \bmod p$. We observe that the two curves in characteristic p defined inhomogeneously by $X^p - Y = 0$ and $X - Y^p = 0$ on the product $D_p \times D_p$ of two projective straight lines are non-singular and of genus zero. Moreover, they intersect transversally at $p^2 + 1$ points, hence the reduction of Φ_p with respect to p is a "reducible curve" with ordinary double points. This being remarked, let Γ' be a reduction of Γ modulo p and assume that Γ' is an irreducible curve of the same genus as Γ. If we project Γ and Γ' generically to a plane, we get the same situation by plane curves. In the same way, if we transform the curve Φ_p by the complete linear system of cubic surfaces, or surfaces of a higher order, of the ambient projective space and then project the image curve generically to a plane, we get the following situation: There are two irreducible plane curves Γ and Φ which are birationally equivalent over a field with a discrete valuation. The reduction Γ' of Γ with respect to this valuation is an irreducible curve. The reduction Φ' of Φ with respect to the same valuation does not contain a straight line as a component and has ordinary double points. We can show, under this circumstance, that Γ' is birationally equivalent to a component of Φ' or it is of genus zero. Since the proof is almost the same as van der Waerden's proof of his Theorem 5 in [13], we shall not repeat it here. At any rate, in the present case, we know that Φ' consists of two curves of genus zero. Therefore Γ', hence also Γ, must be of genus zero. q. e. d.

According to the genus formula for the curve Φ_p [7-III, Theorem 7], there are five exceptional prime numbers, which are 2, 3, 5, 7, 13. Theorem 2 has a number-theoretic consequence, which we shall discuss after this general lemma:

LEMMA 3. *Let γ be a projective curve defined over Q. Then we can construct a non-singular projective model Γ of γ over Q such that its reduction with respect to a prime number p remains non-singular whenever the reduction γ' of γ is genus-preserving,* i. e. *irreducible and has the same genus as γ.*

Proof. Since the proof is similar to the proof of Theorem 1, we shall give only an outline. We note that our lemma could be stated more generally. Let $x = (x_0, x_1, \cdots)$ be a homogeneous generic point of γ over Q, and let S be the integral closure of $Z[x]$ in $Q(x)$. Then S is a finite $Z[x]$-module. Moreover, if S_m is the Z-module of homogeneous elements of S of degree m.

there exists a positive integer m_0 such that $Z[x_i^{-m}S_m]$ is the integral closure of $Z[x_i^{-1}x]$ in its field of fractions for $m \geqq m_0$ and for $x_i \neq 0$ [cf. 9]. On the other hand, since we have $pS \cap S_m = pS_m$, the image S'_m of S_m in S' has the same rang as $Q \otimes S_m$. Let S'' be the integral closure of S' in its field of fractions and let S''_m be its homogeneous part of degree m. Then S'_m is contained in S''_m. If γ' has the same genus as γ, by re-embedding γ, we can assume that $Q \otimes S_m$ and S''_m have the same rank for all m. Then, we get $S' = S''$. Now, as we remarked already, there exists a finite number of elements $\theta_1, \theta_2, \cdots$ of S such that we have $S = Z[x]\theta_1 + Z[x]\theta_2 + \cdots$. Let m_1 be the maximum of maximal degrees of $\theta_1, \theta_2, \cdots$. Then, we can show that $F[S'_m]$ is integrally closed in its field of fractions for $m \geqq m_1$. Thus S_m for $m \geqq \max(m_0, m_1)$ defines a non-singular projective model Γ of γ over Q, and this Γ has the required property. q. e. d.

If we apply this lemma to the curve Φ_n, we see [7-III, Theorem 7] that we can find its non-singular projective model Γ over Q which remains non-singular for every reduction with respect to a prime number not dividing n. Therefore, if Φ_n is of genus one, the absolute invariant of Γ, which we can call the absolute invariant of Φ_n, must be a rational number whose denominator is composed of prime factors of n. If n itself is a prime number p, Theorem 2 shows that p must appear in the denominator. In this way, we get the following.

COROLLARY. *If the curve Φ_p is of genus one, its absolute invariant is a rational number and its denominator is of the form p^e with $e \geqq 1$.*

We know that the curve Φ_p is of genus one only in the three cases $p = 11$, 17, 19. Actually, after some rather complicated calculation, we can show that the corresponding three absolute invariants are

$$-2^{12}31^3 11^{-5} \qquad -3^3 11^3 17^{-4} \qquad -2^{18}7^3 19^{-3}.$$

The reader might think that Theorem 2 and hence the above corollary could be generalized to the case of a non-prime level. However, the situation is more delicate. In fact, the absolute invariants of Φ_n for $n = 14, 15, 20, 21, 24, 27, 32, 36, 49$ [7-III, Theorem 7] are the following:

$5^3 43^3 2^{-6} 7^{-3}$	$13^3 37^3 3^{-4} 5^{-4}$	$2^4 11^3 5^{-2}$	$193^3 3^{-4} 7^{-2}$
$2^4 13^3 3^{-2}$	0	$2^6 3^3$	0	$-3^3 5^3$.

This table shows that not all prime factors of n appear in the denominator. We also note that $2^6 3^3$ and 0 are known as harmonic and equianharmonic cases [cf. 7-III] while, as we can see easily, the last case $-3^3 5^3$ is the case where

the curve has $\frac{1}{2}\cdot(1+(-7)^{\frac{1}{2}})$ as a "complex multiplication." On the other hand, the field of modular functions of level 6 is of genus one and its absolute invariant is zero. In fact, the field admits $LF(2,6)$ as a group of automorphisms, which is an extension of an Abelian group of type $(6,2)$ by a cyclic group of order 6. Since this is the only one metacyclic decomposition of $LF(2,6)$, the curve has sixth roots of unity as complex multiplications [cf. 16]. This shows that we have an equianharmonic case, hence the absolute invariant is zero.

4. Modular correspondences. We shall now discuss modular correspondences and, in particular, the eigen value of modular correspondences of prime degrees operating on differentials of the first kind. As before, let j be transcendental over Q and let A_j be an elliptic curve which is defined over $Q(j)$ and which has j as its absolute invariant. Also, let ω_1 and ω_2 be the generators of the group Ω of points of A_j of order n. We pick a generic point M over Q of the curve C constructed in Section 2 such that we have $Q(M)=Q(j,j_n,Ku(\omega_2))$. Here j_n is the absolute invariant of the factor group of A_j by the cyclic subgroup generated by ω_1. These being recalled, pick a positive integer m which is arbitrary provided it is relatively prime to n. Let j^* be a root of the invariant transformation equation $\Phi_m(X,j)=0$ of degree m and let A_{j^*} be the generic specialization of A_j over the specialization $j\to j^*$ with reference to Q. Then, there exists a homomorphism of A_j to A_{j^*} whose kernel is a cyclic group of order m. Let ω_1^* and ω_2^* be the images of ω_1 and ω_2 under this homomorphism. Since m is relatively prime to n, these images generate the group, say Ω^*, of points of A_{j^*} of order n. Let j_n^* be the absolute invariant of the factor group of A_{j^*} by the cyclic subgroup generated by ω_1^*. We shall show that $(j^*,j_n^*,Ku(\omega_2^*))$ is a generic specialization of $(j,j_n,Ku(\omega_2))$ over Q. Let (ω_1'',ω_2'') be a specialization of (ω_1,ω_2) over the specialization $j\to j^*$ with reference to Q. Then, the specialization is generic and ω_1'' and ω_2'' generate the group Ω^*. Hence, by the Galois theory [7-III, Theorem 3] there exists an automorphism of $Q(j^*,\Omega^*)$ over $Q(j^*)$ which transforms (ω_1'',ω_2'') either to (ω_1^*,ω_2^*) or to $(-\omega_1^*,-\omega_2^*)$. Therefore $(j^*,j_n^*,Ku(\omega_2^*))$ is a generic specialization of $(j,j_n,Ku(\omega_2))$ over Q. Now, let M^* be a specialization of the point M over this specialization. Then M^* is unique and we have $Q(M^*)=Q(j^*,j_n^*,Kp(\omega_2))$. Moreover, from the same Galois theory we conclude that $Q(M,M^*)$ and $Q(j)$ are respectively the compositum and the intersection of $Q(M)$ and $Q(j,j^*)$. The situation remains true even after we extend Q to its algebraic closure Q_0. Therefore $Q(M,M^*)$ is a regular extension of Q and we have

$$[Q(M, M^*) : Q(M)] = [Q(j, j^*) : Q(j)] = m \cdot \prod_{p|m} (1 + p^{-1}).$$

Hence the point $M \times M^*$ of the product $C \times C$ has a locus X over Q. We note that the change of generators ω_1 and ω_2 of Ω amounts to replace the point $M \times M^*$ by its conjugate over $Q(j, j^*)$. Therefore the correspondence X is intrinsically defined. We call X the *modular correspondence* of *degree* m and of *level* n. If we replace M^* by its conjugate M^{**} over $Q(j^*)$, then the point $M \times M^{**}$ so obtained has a locus over the field Q_n of n-th roots of unity over Q. The modular correspondence X is, so-to-speak, singled out from these correspondences, which are sometimes called also modular correspondences.

Now, we shall assume that m is a prime number p. Remember that p does not divide n. Let C_p and X_p be the reductions of C and X with respect to p. Then C_p is a non-singular curve defined over the prime field F by Theorem 1. Let M_p be a generic point of C_p over F. If we denote by $(M_p)^p$ the point of C_p which is obtained by raising the co-ordinates of M_p to their p-th powers, the point $M_p \times (M_p)^p$ of the product $C_p \times C_p$ has a locus I over F. We call I the *Frobenius correspondence* of C_p [cf. 15]. On the other hand, let k be the algebraic closure of F. Then, with the notation of the proof of Lemma 2, we have $k(M_p) = k(j', j_n', Ku(\omega_2')) = k(j', Ku(\Omega'))$. If we write j_n' in the form $j_n(\omega_1')$, there exists an automorphism of $k(M_p)$ over $k(j')$ which transforms $(Ku(\omega_1'), Ku(\omega_2'))$ into $(Ku(\omega_2'), Ku(\omega_1'))$ [7-III, Theorem 3], hence $(j', j_n(\omega_1'), Ku(\omega_2'))$ into $(j', j_n(\omega_2'), Ku(\omega_1'))$. This automorphism transforms M_p to another generic point, say $M_p{}^\tau$, of C_p over F. Let T be the locus of the point $M_p \times M_p{}^\tau$ over k. We note that T is not in general defined over F. At any rate, with these notations, we shall prove the following relation:

LEMMA 4. *If we denote by a right upper prime the effect of the permutation of factors of the product $C_p \times C_p$, we have*

$$X_p = I + T' \circ I' \circ T.$$

In other words, the modular correspondence of degree p splits modulo p into two Frobenius correspondences.

Proof. This important lemma was proved by Eichler [4] in a less general case and later by Shimura [12] in the above form for almost all p. Since our proof is quite similar to Shimura's proof, we shall give only an outline. If p is different from 2, we cover A_j by A_λ such that $A_{\lambda'}$ covers $A_{j'}$. The point is, of course, that $A_{\lambda'}$ is the reduction of A_λ while $A_{j'}$ is not in general the reduction of A_j with respect to p. In case p is 2, we cover A_j and

$A_{j'}$ by A_μ and $A_{\mu'}$ so that $A_{\mu'}$ is the reduction of A_μ with respect to 2. With the aid of these nice coverings, we can show that X_p consists of two parts one of which is I. Let $A_{j''}$ be the generic specialization of $A_{j'}$ over F which is obtained by replacing j' by $(j')^{1/p}$. Let α be a separable homomorphism of $A_{j'}$ to $A_{j''}$ of degree p [cf. 5; see also 8, section 5]. Then the other part of X_p is determined over k by $(j', j_n(\omega_1'), Ku(\omega_2')) \to ((j')^{1/p}, j_n(\alpha\omega_1'), Ku(\alpha\omega_2'))$. On the other hand, there exists an automorphism of $F(j', Ku(\Omega'))$ over $F(j')$ which transforms $(Ku(\omega_1'), Ku(\omega_2'))$ into $(Ku(\omega_1'), Ku(p\omega_2'))$ and its effect on the algebraic closure F_n of F in $F(j', Ku(\Omega'))$ is exactly the p-th power automorphism [7-III, Theorem 3]. Since $F(j', Ku(\Omega'))$ and k are linearly disjoint over F_n, the above automorphism of $F(j', Ku(\Omega'))$ and p-th power automorphism of k give rise to a semi-linear automorphism of $k(j', Ku(\Omega'))$ over k. Therefore, if T^p is the conjugate of T under the p-th power automorphism of k, then T^p is determined by

$$(j', j_n(\omega_1'), Ku(p\omega_2')) \to (j', j_n(\omega_2'), Ku(\omega_1')),$$

hence T is determined by

$$((j')^{1/p}, j_n(\omega_1')^{1/p}, Ku(p\omega_2')^{1/p}) \to ((j')^{1/p}, j_n(\omega_2')^{1/p}, Ku(\omega_1')^{1/p}).$$

However, since we have $j_n(\omega_1')^{1/p} = j_n(p\omega_1')^{1/p} = j_n(\alpha\omega_1')$ and $Ku(p\omega_2')^{1/p} = Ku(\alpha\omega_2')$, we see that $T' \circ I' \circ T$ is the other part of X_p.

The following lemma is the elementary case of the compatibility of Chow's construction of Jacobian varieties [2] and specialization, which we proved some time ago [7-I, Theorem 3].

LEMMA 5. *Let $\mathfrak{o}$ be a discrete valuation ring and let k be its field of fractions. Let Γ be a non-singular projective curve defined over k such that its reduction Γ' with respect to the prime ideal $\mathfrak{p}$ of $\mathfrak{o}$ is again non-singular. Then the reduction J' of the Chow Jacobian variety J of Γ with respect to $\mathfrak{p}$ is the Chow Jacobian variety of Γ'.*

Once we have these two lemmas, by a rather simple argument [4, 12], we can incorporate Weil's results [15, 16] on the eigen values of I operating on the l-primary part of the group of points of finite orders of the Jacobian variety of C_p for a prime number l different from p, and we get the following definitive result:

THEOREM 3. *Let δX be the contravariant map of the vector space of differentials of the first kind on C induced by a modular correspondence X of a prime degree p. Then the eigen values of δX are complex numbers of absolute value at most equal to $2 \cdot p^{\frac{1}{2}}$.*

We note that there is one non-algebraic theorem creeping in the above mentioned "simple argument." In fact, the relation between the representations of the correspondence X on the vector space of differentials of the first kind on C and on the l-primary part of the group of points of finite orders of the Jacobian variety of C can be proved, at least at present, only with the aid of Riemann's transcendental theory.

5. Cusp forms. In this last section, we shall formulate the Petersson conjecture and we shall show that it is solved for dimension -2. Suppose that C is the curve constructed in Section 2. Then any symmetric w-fold differential on C will be called a *modular form* of *dimension* $-2w$ and of *level* n. Here w is, of course, a non-negative integer. It is clear that a modular form can be written uniquely in the form $f(dj)^w$ with some function f on C. A modular form of dimension zero is called a *modular function*. Actually, if we uniformize the curve C by a transcendental parameter z which is a variable in the upper half-plane, the analytic function $f(dj/dz)^w$ is the classical modular form [cf. 6-I]. At any rate, the modular correspondence X of degree m introduced in the previous section gives rise to a contravariant map $T(m)$ of the space of modular forms to itself. The operator $T(m)$ so defined can be called the *Hecke operator* although Hecke put the weight factor m^{w-1} [6-II]. We shall now introduce cusp forms but after this general remark:

Suppose that r is a positive integer and let C^* be the curve in Section 2 for the level rn. Then, the multiplication by r of the elliptic curve A_j defined before induces a unique projection from C^* to C, which is a rational mapping defined over Q. Moreover, if m is relatively prime to r and if X^* is the modular correspondence of degree m and of level rn, the following diagram is commutative:

$$\begin{array}{ccc} & X^* & \\ C^* & \longrightarrow & C^* \\ \Big\downarrow r & & \Big\downarrow r \\ & X & \\ C & \longrightarrow & C. \end{array}$$

The verification is immediate. An attentive reader might have noticed that a similar diagram was used already in the proof of Lemma 4. We note also that in case r is relatively prime to n, the same procedure as above gives rise to a birational correspondence Y of C to itself. If $R(r)$ is the corresponding contravariant map of the space of modular forms, the following multiplication formula comes out rather formally:

$$T(m)\cdot T(m') = \sum_{d|m,m'} d\cdot R(d)\cdot T(mm'/d^2).$$

A modular form $f(dj)^w$ is called a *cusp form* if the corresponding classical modular form $f(dj/dz)^w$ is holomorphic everywhere in the upper half-plane and vanishes at "cusps." As we can see without difficulty, it is the same thing to say that $f(dj)^w$ becomes a modular form of the first kind, i.e. a modular form without poles, on C^* for $r = w$. In particular, a cusp form of dimension -2 and a differential of the first kind are the same thing. At any rate, the space of cusp forms of dimension $-2w$ on C can be identified with a subspace of the space of modular forms of the first kind on C^* for $r = w$. The number of independent cusp forms of a fixed dimension is, therefore, finite and it can be calculated very easily by the Riemann-Roch theorem.

Now in [10] Petersson conjectured that for a subspace of the space of cusp forms of a fixed dimension which is stable under Hecke operators the eigen values of $T(p)$ are complex numbers of absolute value at most equal to $2\cdot p^{\frac{1}{2}}$. It is clear from the above explanation that Theorem 3 solves this conjecture completely for dimension -2. In the case of level one, if we observe that C^* for $r = 6$ is the first curve with positive genus, we see that the first cusp form is of dimenstion -12. Moreover, since we have seen that C^* for $r = 6$ is an equianharmonic elliptic curve, there exists only one cusp form for dimension -12 and it can be expressed as $(dx/y)^6$ on $y^2 = x^3 - 1$. This expression might be helpful for the investigation of the Petersson conjecture in this case, which was raised by Ramanujan.

In concluding this paper, we note that the theory of Hecke operators can be discussed geometrically along our development. We can introduce *integral forms* to be those modular forms which become of the first kind so-to-speak on the limit of C^* for $r \to \infty$. Actually, we can very easily write down the condition and the Riemann-Roch theorem permits us to compute the number of independent integral forms of a fixed dimension. Also, we can prove its stability under $T(m)$, etc. However, we shall leave it as an exercise to the reader.

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THE STRUCTURE OF LOCAL HOMEOMORPHISMS, III.* [1]

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In memory of Aurel Wintner.

In this paper, we continue the study of local homeomorphisms in Euclidean n-space begun in [13] and [14]. The main result to be proved here is that any C^∞ volume preserving transformation defined in some neighborhood of the origin, keeping the origin fixed, whose Jacobian matrix at the origin satisfies a formal condition (**) to be given below, can be brought by a volume preserving change of coordinates to a certain normal form. The set of germs of these normal forms fall into a finite number of classes, each of which constitutes a maximal commutative subgroup of the group of local C^∞ volume preserving maps. Furthermore, the condition (**) will be interpreted as a regularity condition. Thus, we will establish for the group of local C^∞ volume preserving maps a theorem analogous to the one which asserts that any regular element of a real Lie group is conjugate to an element of one of a finite number of Cartan subgroups. A similar analogue for the group of all local C^∞ homeomorphisms was proved in [14]. We shall also prove this result for the group of all local C^∞ maps which preserve volume up to a constant factor, and obtain partial results in this direction for the group for all local C^∞ Hamiltonian maps. Since according to Cartan [4] these are (the local versions of) three of the six primitive infinite groups, it is to be hoped that a general theory of the structure of the infinite Lie groups can be developed. The theorem proved here for the group of local volume preserving maps is the C^∞ and n-dimensional version of a theorem first proved by Moser [9] in the two dimensional analytic case; of also [11].

The 'infinite groups' of Lie and Cartan are not groups in the present day sense. They are families of homeomorphisms defined on various domains of a manifold and satisfying certain partial differential equations. They are closed under the operations of composition and taking inverses whenever

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these operations are defined. The trouble is, of course, that in general, the domains of definition of different homeomorphisms don't match, so they don't form a group. For modern development in the theory of these 'infinite groups' in the large and its application to the problem of equivalence in differential geometry cf. Liberman [8] and Chern [5] and the references cited there. For the foundations of the local theory, cf. Kuranishi [7]. The papers of Kuranishi and Liberman deal only with analytic transformations. The reason for this is that these papers, following Cartan, make essential use of the theory of exterior differential equations; the fundamental existence theorem in the theory of exterior differential equations depends on the Cauchy-Kowalewski theorem. The objects that we have been considering are groups since we consider only transformations leaving the origin fixed, and we identify two transformations if they agree in some neighborhood of the origin; that is, we consider the groups of germs of transformations. In this set-up, in certain respects (mainly the lack of convergence difficulties) the C^∞ case is simpler than the analytic case whereas in other respects the reverse is true. We hope to deal with the analytic case in future publications. Another way to obtain a group is to require that the domain and range of all transformations be a whole manifold. The problems involved in the study of algebraic structure of these groups seem to be quite difficult. Even in the one-dimensional case of homeomorphisms of a circle onto itself, fairly deep diophantine considerations enter, cf. Finzi [6]. These diophantine considerations (in the form of small divisors) also occur in the local analytic theory. It should be observed that the "Poincaré circle problem" is also one of establishing a conjugacy theorem. Here the group is that of all orientation preserving analytic (C^∞) homeomorphisms of the circle onto itself and the group of rotations is a maximal commutative subgroup playing the role of the Cartan subgroup. The problem of determining the correct regularity conditions and proving a conjugacy theorem remains open. It is possible that this problem is a prototype of the study of the algebraic structure of the global infinite Lie groups.

In § 1 of this paper we prove some preparatory lemmas constructing certain maximal abelian subgroups of the group of local C^∞ volume preserving maps and state the main theorem for this group. In § 2 we make a very elementary study of the structure of the algebras of all formal vector fields and formal vector fields with divergence 0. This is done to define the formal conditions (*) and (**) and show that they correspond by analogy to regularity conditions in a Lie group. In § 3 we prove the formal version of the main theorem. The proof given here is a straightforward generalization to n

dimensions of the proof for $n=2$ given in Siegel [11], pp. 135-142. In future publications we shall present a more systematic theory of groups of formal power series transformations and vector fields. The conjugacy theorem proved in § 3 for the formal volume preserving transformations and again in § 9 for the Hamiltonian group will turn out to be special cases of a general theorem on groups of formal power series transformations. We include the proofs given here for these special cases, however, because of their elementary character.

In § 4 we bring every volume preserving transformation satisfying (**) to a preliminary normal form. In § 5 we prove the following theorem. If N and T are two C^∞ homeomorphisms having the same Taylor expansions at the origin and such that no eigenvalue of the Jacobian matrix at the origin is of absolute value 1, then there exists a C^∞ homeomorphism D such as $DTD^{-1}=N$. This theorem, which is essentially the content of Theorem 3 of [14] is reproved here in greater detail because of some additional normalizations, which are necessary for the rest of the argument, on the transformation D. In § 6 we show that every local volume preserving homeomorphism satisfying (**) lies on a one-parameter group. In § 7 we conclude the proof of the conjugacy theorem for the group of local volume preserving maps. In § 8 we prove the corresponding theorem for the group of local maps which preserve volume up to a constant factor. In § 9 we prove a partial result in this direction for the group of Hamiltonian maps.

In what follows, we shall, for the sake of simplicity only consider one of the Cartan subgroups. That is, we will act as if the matrices that arise are real diagonizable, i. e. that the eigenvalues are real. However, all the arguments carry over with the obvious changes to the case of complex eigenvalues. Thus, an expression of the form $x_1x_2\cdots x_n$, might mean $(x_1^2+x_2^2)x_3\cdots x_n$ or $(x_1^2+x_2^2)(x_3^2+x_4^2)\cdots x_n$, etc., depending on how many (pairs of) complex eigenvalues occur. In order not to clutter up the language too much, we frequently do not distinguished between the germ of a transformation and a transformation defined in a sufficiently small neighborhood. The reader is also advised to liberally sprinkle the phrase 'in a sufficiently small neighborhood of' throughout the paper. This paper depends fairly heavily on the results of [13] and the ideas of [14]. It can be read independently of [14], but not of [13].

1. Before proceeding to the statement of the theorem let us briefly resume and place in a more algebraic setting the results of [14]. Let $\boldsymbol{C}^n$

denote the group of all local C^∞ maps. Then the set of all (complex) diagonal linear transformations of $\boldsymbol{G}^n$, i.e., transformations of the form

(i) $$x_i^* = \lambda_i x_i$$

constitute a commutative subgroup of $\boldsymbol{G}^n$. (It is to be understood that (i) is a real transformation. If a complex eigenvalue, λ_i, occurs in (i) then so will $\bar{\lambda}_i$ and the resulting pair of equations in (i) represents the corresponding rotation-dilatation in the appropriate two dimensional space. This and/or a similar convention will be followed throughout this paper.) We observe that the transformations (i) actually form a maximal commutative subgroup of $\boldsymbol{G}^n$. In fact, if T commutes with all of (i), then since projection onto the Jacobian matrix is a homomorphism and the (i) form a maximal commutative subgroup of the general linear group, the linear terms of T must be of the form (i). Thus T can be written in the form

(ii) $$x_i^* = \mu_i x_i + \phi_i(x_1, \cdots, x_n),$$

with $\phi_i(0, \cdots, 0) = O(v^2)$. The condition that T commutes with all of (i) implies that $\lambda_i^{-1}(\lambda_1 x_1, \cdots, \lambda_n x_n) = \phi_i(x_1, \cdots, x_n)$ for all non-vanishing λ's. Letting the λ's tend to zero shows that $\phi_i = 0$. The main theorem of [10] asserts that if S is an element of $\boldsymbol{G}^n$ whose matrix of linear terms is equivalent to a transformation of the form (i) and such that

(*) $$\lambda_i \neq \lambda_1^{m_1} \cdots \lambda_n^{m_n} \qquad (m_j \text{ non-negative integers with } \sum m_i > 1),$$

then S is conjugate via an inner automorphism of $\boldsymbol{G}^n$ to an element of (i). We shall discuss condition (*) further in § 2.

Now let $\boldsymbol{V}^n$ denote the group of all volume preserving maps. Here, the projection onto linear terms maps $\boldsymbol{V}^n$ onto the special linear map. A maximal commutative subgroup of the special linear group is the set of all transformations of the form

(1) $$x_i^* = \lambda_i x_i \text{ with } \prod \lambda_i = 1.$$

The transformations (1) do not, however, form a maximal commutative subgroup of $\boldsymbol{V}^n$. In fact, any transformation of the form (1) clearly commutes with any transformation of the form

(2) $$x_i^* = x_i f_i(x_1 x_2 \cdots x_n);$$

Here the f_i are functions of one variable, the product $x_1 \cdots x_n$. It is easy to see that any transformation T of $\boldsymbol{V}^n$ commuting with all of (1) must be of the form (2); by the same argument as before, T has the form (ii) with

$\prod \mu_i = 1$. Now if $\phi_i = x_i\psi_i + \eta_i(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots x_n)$, then, letting the $\lambda_j \neq \lambda_i$ approach 0, we can conclude that $\eta_i = 0$ and $\psi_i(\lambda_1 x_1, \cdots, \lambda_n x_n) = \psi_i(x_1, \cdots, x_n) =$ for all λ_i such that $\prod \lambda_i = 1$. If one of the x_i's vanishes we can apply the previous argument so that the $\psi_i = 0$ along the hyperplanes $x_j = 0$. If none of the x_i vanishes and $x_1 \cdots x_n = y_1 \cdots y_n$, then taking $\lambda_i = y_i/x_i$ we have $\psi_i(x_1, \cdots, x_n) = \psi_i(y_1, \cdots, y_n)$. Thus T has the form (2).

Of course, not all transformations of the form (2) belong $\boldsymbol{V}^n$. In order to determine which ones do, we need

LEMMA 1. *Let T be a transformation of the form* (2) *and let $\omega = x_1 \cdots, x_n$. Then the Jacobian of T is $d/d\omega(\omega \prod f_i(\omega))$.*

Proof. A direct computation shows that the transformation S given by

$$\begin{aligned} x_i^* &= x_i && (i = 1, \cdots, n-2), \\ x_{n-1}^* &= x_{n-1} f_n(\omega), \\ x_n^* &= x_n / f_n(\omega) \end{aligned} \tag{3}$$

is volume preserving. The transformation TS has the same Jacobian as T and has the form

$$\begin{aligned} x_i^* &= x_i f_i(\omega) && (i = 1, \cdots, n-2), \\ x_{n-1}^* &= x_{n-1} f_{n-1}(\omega) f_n(\omega), \\ x_n^* &= x_n. \end{aligned} \tag{4}$$

Continuing in this manner, we finally arrive at a transformation given by

$$\begin{aligned} x_1^* &= x_1 \prod f_i(\omega), \\ x_i^* &= x_i && (i > 1). \end{aligned} \tag{5}$$

A direct computation of the Jacobian of (5) proves the lemma.

In particular, setting $h(\omega) = \prod f_i(\omega)$, if T is to be a volume preserving transformation, h must satisfy the differential equation $\omega h'(\omega) + h(\omega) = 1$. If h is to be regular at the origin, we must have $h = 1$. Since all transformations (2) with $h = 1$ commute with one another and the inverse of (2) is again of that form, we have proved

LEMMA 2. *The set of all transformations of the form* (2) *with $\prod f_i = 1$ is a maximal commutative subgroup of $\boldsymbol{V}^n$. We will denote this subgroup by $\boldsymbol{C}^n$.*

If we examine the formal condition (*), we see that it is violated for

every element of $\boldsymbol{V}^n$ since $\prod \lambda_i = 1$. By Theorem 3 of [14], if (for a transformation $T \in \boldsymbol{V}^n$) this is the only violation of (*), there exists a C^∞ change of coordinates R such that RTR^{-1} is of the form (2). The purpose of this paper is to show that R can be chosen to be volume preserving so that (automatically) RTR^{-1} is an element of $\boldsymbol{C}^n$. Thus we wish to prove

THEOREM 1. *Let $T \in \boldsymbol{V}^n$, $n > 2$, have a (complex) diagonizable linear part whose eigenvalues, $\lambda_1, \cdots, \lambda_n$, satisfy*

(**) *if $\lambda_i = \lambda_1^{m_1} \cdots \lambda_n^{m_n}$, then $m_i - 1 = m_j$ for all $j \neq i$,*

then there exists an $R \in \boldsymbol{V}^n$ such that $RTR^{-1} = N \in \boldsymbol{C}^n$. For the case $n = 2$, the above statement is true under the additional hypothesis $|\lambda_i| \neq 1$.

Remark. In the case $n > 2$, the condition $\lambda_i \neq 1$ is a consequence of (**).

2. In this section, which will not be needed for what follows, we shall try to give some meaning to conditions (*) and (**). We do this by examining the root structure of the Lie algebras of all formal fields and formal vector fields with divergence zero. In future publications we shall examine the connection between groups of formal power series transformations and Lie algebras of formal vector fields and obtain a structure and representation theory for such algebras. For the present, our justification of (*) and (**) will merely be by analogy to the case of a finite dimensional Lie group.

By a formal vector field, f, we shall mean a vector $f = (u_1 \cdots u_n)$ where each of the u_i is a formal power series in $x_1, \cdots, x_n$ with no constant terms. For simplicity, we shall allow complex coefficients. By the Lie bracket $[g, f]$, with $g = (v_1 \cdots v_n)$, we shall mean the formal vector field $h = (w_1, \cdots, w_n)$, where the w_i are given by

$$w_i = \sum_j (u_j \partial v_i / \partial x_j - v_j \partial u_i / \partial x_j). \tag{6}$$

Let $\boldsymbol{g}^n$ denote the collection of all formal vector fields and let $\boldsymbol{v}^n$ denote those vector fields $f = (u_1, \cdots, u_n)$ such that $\sum \partial u_i / \partial x_i = 0$. Then it is clear that $\boldsymbol{g}^n$ and $\boldsymbol{v}^n$ are (infinite dimensional) Lie algebras. Let us first examine the algebra $\boldsymbol{g}^n$. The set of diagonal linear vector fields, $\boldsymbol{l}^n$, forms a commutative subalgebra. Let $l = (\lambda_1 x_1, \cdots, \lambda_n x_n)$ be an element of $\boldsymbol{l}^n$ and let $f = (u_1, \cdots, u_n)$ be in $\boldsymbol{g}^n$, where

$$u_i = \sum a^i{}_{j_1 \cdots j_n} x_1^{j_1} \cdots x_n^{j_n}. \tag{7}$$

Then $[l, f] = (w_1, \cdots, w_n)$, where the w are given by

$$w_i = \sum(\lambda_i - j_1\lambda_1 \cdots - j_n)a^i{}_{j_1\cdots j_n}x_1{}^{j_1}\cdots x_n{}^{j_n}. \tag{8}$$

From (8) it follows immediately that $\boldsymbol{l}^n$ is a maximal abelian subalgebra of g^n. Furthermore, the linear forms $\lambda_i - \sum m_j\lambda_j$ (m_j non-negative integers with $\sum m_i \geqq 1$) form a system of roots on this commutative subalgebra. The corresponding root vectors are the vector fields $r = (r_1, \cdots, r_n)$, where $r_k = a^k x_1{}^{n_{k_1}}x_2{}^{n_{k_2}}\cdots x_n{}^{n_{k_n}}$ and the integers n_{k_i} satisfy $n_{k_l} - \delta_k{}^l = m_l - \delta_i{}^l$. Here it is to be understood that if no non-negative n_k with $\sum n_{k_i} > 1$ can be found, the term r_k is to be taken as zero.

Thus the condition (*) is the straightforward generalization of the regularity condition for an element in a semi-simple Lie group. (The reason that in (*) we require $\sum m_i > 1$ instead of $\sum m_i \geqq 1$ is that in the hypothesis of Theorem 1 of [10] we assume that the linear terms of the transformation is diagonizable.) Thus Theorem 1 of [10] asserts that any regular element of G^n is conjugate to an element of the "Cartan subgroup" of diagonal linear transformations.

In the algebra $\boldsymbol{v}^n$ the diagonal linear vector fields of trace zero form a commutative subalgebra. The linear forms $\lambda_i - \sum m_j\lambda_j$ except for those where $m_j - \delta_j{}^i = m$, form a system of roots. The vector fields of the form $(r_1, \cdots, r_n)$, where the $r_i = x_i f_i(\omega)$ and $\sum f_i = 0$, form a maximal commutative subalgebra. Theorem 1 thus asserts (at least for $n > 2$) that every regular element of $\boldsymbol{V}^n$ is conjugate to an element of the Cartan subgroup $\boldsymbol{C}^n$.

3. In this section we prove the formal analogue of Theorem 1. The proof is a straightforward generalization of the classical two-dimensional case, cf. Siegel [7].

THEOREM 2. *Let $\mathfrak{G}^n$ denote the group of formal volume preserving transformations and $\mathfrak{C}^n$ the formal transformations of the form* (2), *where $\prod f_i(\omega) = 1$. Then if $T \in \mathfrak{G}^n$ satisfies* (**), *there exists an $R \in \mathfrak{G}^n$ such that*

$$RN = TR, \text{ where } N \in \mathfrak{C}^n. \tag{9}$$

Proof. By a preliminary linear change of variables we can assume that the linear part of T is a diagonal form. We shall then look for an R whose linear part is the identity matrix. Then if $T = (t_1, \cdots, t_n)$, where

$$t_i = \lambda_i x_i + \sum_{(j_1+\cdots+j_n>1)} t^i{}_{j_1\cdots j_n}x_1{}^{j_1}\cdots x_n{}^{j_n}, \tag{10}$$

we wish to find an $R = (r_1, \cdots, r_n)$ with

$$r_i = x_i + \sum_{(j_1+\cdots+j_n>1)} r^i{}_{j_1\cdots j_n} x_1{}^{j_1}\cdots x_n{}^{j_n} \tag{11}$$

and an $N = (x_1 f_1(\omega), \cdots, x_n f_n(\omega))$, where the f_i are given by

$$f_i = \sum_{l=0}^{\infty} f^i{}_l \omega^l = \sum_{l=0}^{\infty} f^i{}_{l\cdots l} x_1{}^l \cdots x_n{}^l. \tag{12}$$

(Let us repeat that we are proceeding as if the eigenvalues λ_i are real. The transition to complex eigenvalues can be made either by passing to the complex field and then observing that all the formal power series that arise satisfy a reality condition as was done in [13], cf. also [3] and [10]; or by staying within the real field and making only trivial modifications of the arguments below.)

We now follow § 4 of [13]. Given a lattice vector $(\alpha_1, \cdots, \alpha_n)$, we define its height k to be $\sum \alpha_k$. Then, substituting (10), (11) and (12) into (9) and comparing the coefficients of $x_1{}^{\alpha_1}\cdots x_n{}^{\alpha_n}$, we obtain

$$r^i{}_{\alpha_1\cdots\alpha_n}\lambda_1{}^{\alpha_1}\cdots\lambda_n{}^{\alpha_n} + f^i{}_{l\cdots l} + P_1 = \lambda_i r^i{}_{\alpha_1\cdots\alpha_n} \tag{13}$$

if $\alpha_k - \delta_i{}^k = l$. In all other cases we obtain

$$r^i{}_{\alpha_1\cdots\alpha_n}\lambda_1{}^{\alpha_1}\cdots\lambda_n{}^{\alpha_n} + P_1 = \lambda_i r^i{}_{\alpha_1\cdots\alpha_n} + P_2. \tag{14}$$

In both equations (13) and (14) the P's represent polynomials in $r^j{}_{\beta_1\cdots\beta_n}$ and $f^j{}_{p\cdots p}$, where the $(\beta_1, \cdots, \beta_n)$ and $(p, \cdots, p)$ are all of height less than k. Thus, if the case (14) applies, $r^i{}_{\alpha_1\cdots\alpha_n}$ is uniquely determined by the $r^j{}_{\beta_1\cdots\beta_n}$ and $f^j{}_{p\cdots p}$ of lower height by virtue of condition (**). If the case of (13) applies, no condition at all is imposed on $r^i{}_{\alpha_1\cdots\alpha_n}$ (since $\prod\lambda_i = 1$), whereas the $f^1{}_{l\cdots l}$ is determined by the terms of lower height. If we were to choose the $r^i{}_{\alpha_1\cdots\alpha_n}$ corresponding to the case (13) arbitrarily, we would have an R of the form (11) and an N of the form (12) which satisfy (9). However, this will not do for Theorem 2 since we want: a) R to be volume preserving and b) $\prod f_i = 1$. Now b) is a consequence of a) by (the formal analogue of) Lemma 2. Thus we must show that it is possible to choose the $r^i{}_{\alpha_1\cdots\alpha_n}$ at our disposal in such a way that

$$\Delta(x_1, \cdots, x_n) = |\partial r^i/\partial x^j| = 1. \tag{15}$$

We do this by making the following choice of $r^i{}_{\alpha_1\cdots\alpha_n}$: Choose the $r^i{}_{\alpha_1\cdots\alpha_n}$, where $\alpha_j - \delta_i{}^j = q$, so that

$(16)_q$ $\quad \Delta - 1$ contains no term of the form ω^p with $p \leqq q$.

We first show that a choice is possible. For the case $q = 0$, this condition is

trivial. Now in the power series expansion of Δ the coefficient of ω^q is a polynomial in those $r^i{}_{\alpha_1 \cdots \alpha_n}$ of height at most $nq+1$. Thus if $(16)_q$ is satisfied for $q < l$, then we must choose the $r^i{}_{\alpha_1 \cdots \alpha_n}$, where $\alpha_j - \delta_i{}^j = l$, so that in the formal power series expansion of $\Delta - 1$ the coefficient of ω^l is 0. Remembering that the linear part of R is the identity matrix, we obtain that the coefficient of ω^l is

$$r^1{}_{l+1,l\cdots l} + \cdots + r^n{}_{l\cdots l,l+1} + Q, \tag{17}$$

where Q is a polynomial in terms of height $\leqq nl$. Thus the condition $(16)_q$ reduces to a single linear equation for the $r^i{}_{\alpha_1 \cdots \alpha_n}$ and so can be satisfied.

We now choose the $r^i{}_{\alpha_1 \cdots \alpha_n}$ with $\alpha_j - \delta_i{}^j = q$ arbitrarily, subject to the condition $(16)_q$. Then R and N are uniquely determined by (13) and (14). Furthermore, $\Delta - 1$ contains no term in ω. We wish to show that (15) is satisfied. Now since $RN = TR$ and N is of the form (2), we have

$$\Delta(x_1 f_1, \cdots, x_n f_n)\, d/d\omega[\omega \textstyle\prod f_i(\omega)]) = \Delta(x_1, \cdots, x_n) \tag{18}$$

since T is volume preserving and the Jacobian of N is given by (the formal analogue of) Lemma 1. If we set

$$\Gamma(x_1, \cdots, x_n) = \sum \gamma_{\alpha_1 \cdots \alpha_n} x_1{}^{\alpha_1} \cdots x_n{}^{\alpha_n} = \Delta(x_1 \cdots x_n) - 1$$

and

$$h(\omega) = \sum h_{ll\cdots l} x_1{}^l \cdots x_n{}^l = d/d\omega[\omega \textstyle\prod f_i(\omega)] - 1,$$

then $\Gamma(0, \cdots, 0) = h(0) = 0$. We wish to prove that Γ and h vanish identically. In terms of Γ and h, (18) becomes

$$\Gamma(x_1 f_1, \cdots, x_n f_n) + h(\omega) + \Gamma(x_1 f_1, \cdots, x_n f_n) h(\omega) = \Gamma(x_1, \cdots, x_n). \tag{19}$$

We have chosen R so that the $\gamma_{ll\cdots l} = 0$. We prove that the remaining $\gamma_{\alpha_1 \cdots \alpha_n}$ and $h_{ll\cdots l}$ vanish by induction on their height. As we have seen, the terms of height 0 vanish. We thus assume that all terms of height $k-1$ vanish. If $(\alpha_1, \cdots, \alpha_n)$ is of height k and is not of the form $(l, l, \cdots, l)$, then comparing the coefficients of $x_1{}^{\alpha_1} \cdots x_n{}^{\alpha_n}$ in (19), we obtain (since $f^i{}_{00\cdots 0} = \lambda_i$)

$$\gamma_{\alpha_1 \cdots \alpha_n} \lambda_1{}^{\alpha_1} \cdots \lambda_n{}^{\alpha_n} + Q = \gamma_{\alpha_1 \cdots \alpha_n},$$

where Q is a polynomial with no constant terms in the terms of lower height; thus $Q = 0$. Then, by (**), $\gamma_{\alpha_1 \cdots \alpha_n} = 0$. Comparing the coefficients of $x_1{}^l x_2{}^l \cdots x_n{}^l$, we obtain, since $\gamma_{ll\cdots l} = 0$, that $h_{ll\cdots l}$ is a polynomial in the lower order terms. Thus $h_{ll\cdots l} = 0$. This completes the proof of Theorem 2.

4. In this section we show how to bring every volume preserving transformation to a preliminary normal form. For this purpose we need the analogue of Lemma 5 of [10].

LEMMA 3. *Let $S=(s_1,\cdots,s_n)$ be a formal volume preserving transformation whose linear part is the identity matrix. Then there exists a C^∞ volume preserving transformation $S=(s_1,\cdots,s_n)$, where s_i is the Taylor series expansion of s_i.*

Proof. Choose $s_1,\cdots,s_{n-1}$ arbitrarily as C^∞ functions whose Taylor series is $s_1,\cdots,s_{n-1}$. (This can be done by Lemma 5 of [10].) Let s_{n0} denote the formal power series in $n-1$ variables obtained from s_n by setting $x_n=0$, and let s_{n0} denote a C^∞ function defined on the hyperplane $x_n=0$ and whose Taylor series expansion is s_{n0}. Then the linear partial differential equation $|\partial s_i/\partial x_j|=1$ for s_n is non-characteristic on the hyperplane $x_n=0$. On this hyperplane we assign the initial values $s_n(0)=s_{n0}$. Then the solution s_n is a C^∞ function which, together with $s_1,\cdots,s_{n-1}$, gives a volume preserving transformation S. Since the corresponding formal partial differential equation has the unique solution s_n, the Taylor series of s_n is s_n, proving the lemma. If S is of the form (2), then we can prove the lemma more directly and still retain the form (2). In fact, if $s_i=x_if_i(\omega)$, then choose the first $n-1$ C^∞ functions f_i of one variable ω and set $f_n=1/f_1\cdots f_{n-1}$.

Using Theorem 2 and Lemma 3, we obtain

LEMMA 4. *Let T be an element of $\boldsymbol{V}^n$ satisfying (**). Then there exists an $N\in\boldsymbol{C}^n$ and a volume preserving change of coordinates so that in the new coordinate system $T-N$ vanishes at the origin with infinite order.*

Now since no $|\lambda_i|=1$,, we can assume that the $|\lambda_i|$ are ordered so that

$$|\lambda_1|\leqq\cdots\leqq|\lambda_k|<1<|\lambda_{k+1}|\leqq\cdots\leqq|\lambda_n|.$$

As in [10], we introduce the norms

$$\|x\|_+=\sum_1^k x_i^2,\qquad \|x\|_-=\sum_{k+1}^n x_i^2,\qquad \|x\|=\|x\|_++\|x\|_-$$

and the

$$S_+=\{x\mid\|x\|_+<\|x\|_-\},\qquad S_-=\{x\mid\|x\|_+>\|x\|_-\},$$

$$I_+=\{x\mid\|x\|_+=0\},\ C=\{x\mid\|x\|_+=\|x\|_-\}\ \text{and}\ I_-=\{x\mid\|x\|_-=0\}.$$

Now the spaces I_+ and I_- are invariant under N, and N is an expansion (contraction) on I_+ (I_-). By Theorem 7 of [13], there exist two invariant surfaces under T,

$$J_+\colon x_i=\phi_i(x_{k+1},\cdots,x_n)\qquad (i=1,\cdots,k),$$

$$J_-\colon x_j=\phi_j(x_1,\cdots,x_k)\qquad (j=k+1,\cdots,n),$$

where the ϕ_i are C^∞ and vanish at the origin with infinite order. Now the transformation

$$x^*_i = x_i - \phi_i(x_{k+1}, \cdots, x_n) \text{ for } i \leqq k, \; x^*_i = x_i \text{ for } i > k$$

is clearly volume preserving and is the identity at the origin up to terms of infinite order. In the x^* coordinate system the surface J_+ coincides with I_+ and J_- has the form

$$x^*_j = \eta_j(x^*_1, \cdots, x^*_k), \qquad j > k,$$

where the η vanish with infinite order at the origin. Then the transformation

$$x^{**}_i = x^*_i \text{ for } i \leqq k, \qquad x^{**}_j = x^*_j - \eta_j(x^*_1, \cdots, x^*_k) \text{ for } j > k$$

leaves I_+ invariant and takes J_- into I_-. We thus can choose an $M \in \boldsymbol{V}^n$ so that the invariant surfaces of MTM^{-1} and N are I_+ and I_-. The new T and N are expansions on I_+ which coincide up to terms of infinite order of the origin. We can, by Theorem 2 of [9], find a C^∞ transformation X defined on the space I_+ so that

$$XT_{I_+}X^{-1} = N_{I_+}.$$

If x_+ and x_- denote the projections of x onto I_+ and I_- respectively, we then defined a transformation Q in a neighborhood of the origin by

$$x^*_+ = Xx_+, \qquad x^*_- = (\det X(x_+))^{-1/(n-k)}x_-.$$

Then $Q \in \boldsymbol{V}^n$, the restriction of Q to I_- is the identity and $Q - E$ vanishes at the origin with infinite order, where E is the identity transformation. After transforming by Q, the transformations T and N coincide on I_+. Doing the same for I_-, we obtain

LEMMA 5. *Let $T \in \boldsymbol{V}^n$ satisfy (**). Then there exist an $N \in \boldsymbol{C}^n$ and an $S \in \boldsymbol{V}^n$ so that $STS^{-1} - N$ vanishes along I_+ and I_- and vanishes at the origin with infinite order.*

5. In this section, we shall construct a $D \in \boldsymbol{G}^n$ so that

$$DT = ND. \tag{20}$$

This is a consequence of Theorem 3 of [14]. However, since no details of the proof are given there, we shall reprove this theorem here. For a sketch of and a motivation for the arguments of this section cf. [14], pp. 624-625.

We shall assume for convenience that N has been extended to all space so as to remain a C^∞ transformation (not necessarily a homeomorphism) and

so that its derivatives of every order remain bounded. We choose a ball B_ρ: $\| x \| \leqq \rho$ so small that $\| (T-L)x \|_z < \epsilon \| x \|_z$, where L is the linear part of T, which we can do since I_+ and I_- are invariant. Then $T(B_\rho \cap S_+) \subset S_+$ and we set, for $r < \rho/3$

$$
\begin{aligned}
W_{0,r} &= \overline{B_r \cap S_+ - T(B_\rho \cap S_+)}, \\
W_{n,r} &= TW_{n-1,r} \cap B_r \text{ for } n > 0, \text{ and} \\
W_{n,r} &= T^{-1}W_{n+1,r} \cap B_r \text{ for } n < 0.
\end{aligned} \tag{21}
$$

It is then clear as in the proof of Lemma 2 of [10], that

$$
B_{r/3} \subset I_+ \cup I_- \cup \bigcup_{n=-\infty}^{\infty} W_n. \tag{22}
$$

We define the transformation D in $W = W_{0,r}$ as follows: we set $D =$ identity on C and (for later purposes) in a neighborhood in W of every point of C different from the origin, where at each point we choose a neighborhood so small that it does not intersect TC. This then defines D on TC by (20). That is, at a point $x = Ty$ of TC, we set $Dx = Ny$. Then, as in Lemma 8 and the paragraphs immediately following it of [14], we can "fill in" D so that $D - E$ vanishes at the origin with infinite order, where E is the identity transformation, and satifies the above requirements at the boundary of C. Now D is defined on $W_{0,r}$. We define it on all $W_{k,r/3}$ by (20). That is, for $x \in W_{k,r}$ we set $Dx = N^k D T^{-k} x$. Then D is defined everywhere in a neighborhood of the origin except on the surfaces I_+ and I_-, and is a C^∞ at these surfaces. We first show that it is continuous there. For this purpose it is sufficient to show that $D - E$ evaluated at any point of $W_{k,r/3}$ tends to zero as $|k| \to \infty$. We shall show this for $k \to +\infty$; for $k \to -\infty$ the situation is entirely analogous. We first need a slightly different version of Lemma 3 of [10].

LEMMA 6. *There exists a constant $\kappa < 1$ such that for sufficiently small r and all sufficiently large $k \geqq k_0$,*

$$
T^{-1}W_{k+1,r} \subset W_{k,\kappa r}. \tag{23}
$$

Proof. Since I_+ and I_- are invariant we can choose r_1 so small that $\| Tx \|_- \geqq c_1 \| x \|_-$ and $\| Tx \|_+ \leqq c_2 \| x \|_+$ for all $x \in B_r$, where $c_1 > 1 > c_2$. We then choose a k_δ so large that $\| x \|_- \geqq (1-\delta) \| x \|$ for all $x \in W_{k,r}$, where $k \geqq k_\delta$, and $r \leqq r_1$. Then $\| Tx \| \geqq \| Tx \|_- \geqq c_1 \| x \|_- \geqq c_1(1-\delta) \| x \|$ for $x \in W_{k,r}$. If we choose δ so small that $c_1(1-\delta) > 1$, then setting $\kappa = 1/c_1(1-\delta)$, we have (23).

For any $l < k_0$, $\| T^k x \| \geqq \| T^l x \|_- \geqq c_1{}^k \| x \|_- \geqq \frac{1}{2} c_1{}^k \| x \|$, so that $\| x \|$ is decreased by at most a factor of $\frac{1}{2}$.

We also need an analogue of the mean value theorem.

LEMMA 7. *Let U, Y, and Z be three smooth transformations (with the proper domains of definition), $U(x) = (u_1(x_1, \cdots, x_n), \cdots, u_n(x_1, \cdots, x_n))$, etc. Then*

$$U(Z+Y) = U(Z) + U' \cdot Y, \tag{24}$$

where U' is a matrix made up of the partial derivatives $\partial u_i / \partial x_j$ evaluated at various points on the lines joining $(z_1, \cdots, z_n)$ to

$$(z_1 + y_1, \cdots, z_n), \cdots, (z_1 + y_1, \cdots, z_{n-1} + y_{n-1}, z_n)$$

to $(z_1 + y_1, \cdots, z_n + y_n)$.

Proof.

$$\begin{aligned} u_i(z_1 + y_1, \cdots, z_n + y_n) = u_i(z_1, \cdots, z_n) &+ (u_i(z_1 + y_1, \cdots, z_n) \\ &- u_i(z_1, \cdots, z_n)) + \cdots + (u_i(z_1 + y_1, \cdots, z_n + y_n) \\ &- u_i(z_1 + y_1, \cdots, z_{n-1} + y_{n-1}, z_n)). \end{aligned}$$

Then apply Taylor's formula.

If we now set

$$N = T + G \tag{25}$$

and

$$P_k = N^k D T^{-k} - E \text{ on } W_k, \tag{26}$$

then, since G and P_0 vanish at the origin with infinite order, for any q there exists a constant A_q such that

$$\| G(x) \| < A_q \| x \|^q \tag{27}$$

and

$$\| P_0(x) \| < A_q \| x \|^q. \tag{28}$$

In view of (26), we have the following recursion formula for P_k:

$$P_{k+1} = N(P_k + E) T^{-1} - E. \tag{29}$$

By repeated use of Lemma 7, we can rewrite (29) as

$$P_{k+1} = N'(E + \theta P_k) \cdot P_k T^{-1} + G T^{-1}, \tag{30}$$

where the matrix N' is the one given by Lemma 7. Thus

$$P_k = GT^{-1} + N'(E+\theta P_{k-1})GT^{-2} + N'(E+\theta P_{k-1})N'(E+\theta P_{k-2})GT^{-3} + \cdots + N'(E+\theta P_{k-1})\cdots N'(E+\theta P_0)P_0T^{-k}. \tag{31}$$

Now set

$$M = \sup_{B_r} |\, \partial n_i/\partial x_j \,| \tag{32}$$

and choose q so large that

$$\kappa^q M n = \alpha < 1. \tag{33}$$

In view of Lemma 6 and (27), for any point in $W_{k,r}$, we have

$$\sup_{W_{k,r}} \| GT^{-p}(x) \| \leqq A_q r^{q_\kappa qp}, \tag{34}$$

for $k - p \geqq k_0$, and

$$\sup_{W_{k,r}} \| GT^{-p}(x) \| \leqq A_q 2^{q_\gamma q_\kappa q(k-k_0)} \tag{35}$$

for $k - p < k_0$. Similarly, by (28),

$$\sup_{W_{k,r}} \| P_0T^{-k}(x) \| \leqq A_q 2^{q_\gamma q_\kappa q(k-k_0)}. \tag{36}$$

Using (31), we have, for large values k and all m such that $k - m \geqq k_0$,

$$\| P_k \| \leqq \| GT^{-1} + N'(E+P_{k-1})GT^{-2} + \cdots + N'(E+P_{k-1})\cdots \cdots N'(E+P_{k-m})GT^{-m} \| + A_q r^q \alpha^m/(1-\alpha) + 2^q k_0 (Mn)^{k_0} A_q \alpha^{k-k_0} \tag{37}$$

Now for any ϵ, we choose m_ϵ so large that the second term on the right of (37) is $\leqq \epsilon/3$. We then take k_ϵ so large that the third term on the right $\leqq \epsilon/3$ for all $k \geqq k_\epsilon$. The remaining term on the right can be majorized by an expression of the form

$$B_2 \| GT^{-1} \| + B_2 \| GT^{-2} \| + \cdots + B_m \| GT^{-m} \|, \tag{38}$$

where the B's are constants. Now as $k \to \infty$, x approaches I_+, as do the points $T^{-1}x, \cdots, T^{-m}x$. (Remember that now m is large but *fixed.*) Since, by assumption $G(p) \to 0$ as $p \to I_+$, by choosing $k_{-\epsilon}$ sufficiently large, we can make (38) $< \epsilon/3$. We have thus proved the continuity of D at I_+.

The sequence $(1/\| x \|^h)P_m$ also converges (to zero) by the above argument. This can be seen most easily by absorbing the h into the q of (27) and (28). Thus for any h, there exists a constant A'_n such that

$$\| (D-E)x \| \leqq A'_n \| x \|^h. \tag{39}$$

In order to prove that D is C^∞ at I_+, it suffices to prove

LEMMA 8. *Let* $D - E = F = (f_1, \cdots, f_n)$. *Then if* $p \in I_+$, $p_k \notin I_+$, *and* $p_n \to p$, *the sequence* $\partial f_i(p_k)/\partial x_1^{l_1} \cdots \partial x_n^{l_n}$ *converges.*

The proof of Lemma 7 is essentially the same as the proof of the continuity of D except that the estimates are a little more complicated. In the wedge W_k, the transformation F coincides with P_k. Thus if f_i^k are the coordinate functions of P_n, (29) becomes

$$f_i^{k+1}(x) = n_i(T^{-1}x + P_k T^{-1}x) - x_i. \tag{40}$$

Differentiating (40) and setting $T^{-1} = (t^*_1, \cdots, t^*_n)$, we obtain

$$\begin{aligned}(\partial f_i^{k+1}/\partial x_j)(x) &= [\sum_{j_1}(\partial t_i/\partial x_{j_1})(T^{-1}x + P_k T^{-1}x)(\partial t^*_{j_1}/\partial x_j)(x) - \delta^j{}_i] \\ &+ [\sum_{j_1}(\partial q_i/\partial x_{j_1})(T^{-1}x + P_k T^{-1}x)(\partial t^*_{j_1}/\partial x_j)(x)] \\ &+ [\sum_{j_1, j_2}(\partial u_i/\partial x_{j_1})(T^{-1}x + P_k T^{-1}x)(\partial f^k{}_{j_1}/\partial x_{j_2})(T^{-1}x)(\partial t^*_{j_2}/\partial x_j)].\end{aligned} \tag{41}$$

If we apply Taylor's formula (as in Lemma 5), we have

$$(\partial t_i/\partial x_{j_1})(T^{-1}x + P_k T^{-1}x)(\partial t_i/\partial x_{j_1})(T^{-1}x) + \sum(\partial t_i/\partial x_{j_1}\partial x_l)(T^{-1}x + \theta_l P_k)f_l(Tx^{-1}).$$

Since $\sum(\partial t_i/\partial x_{j_1})(T^{-1}x)(\partial t^*_{j_1}/\partial x_y) = \partial x_i(TT^{-1}x)/\partial x_j = \delta^j{}_i$, (41) can be written as

$$\partial f_i^{k+1}/\partial x_j = \sum_{j_1, m_1} A_{ij_1}(\partial f^k{}_{j_1}/\partial x_{m_1})(T^{-1}x)B_{m_1} + P(i,j), \tag{42}$$

where

$$A_{ij} = (\partial n_i/\partial x_j)(T^{-1}x + P_k T^{-1}x), \tag{43}$$

$$B_{ij} = (\partial t^*_i/\partial x_j)(x), \tag{44}$$

and $P(i,j)$ is a polynomial in $\partial n_{j_1}/\partial x_{j_2}$, $\partial^2 n_{j_1}/\partial x_{j_2}\partial x_{j_3}$, $\partial t_{j_1}/\partial x_{j_2}$, $\partial g_l(T^{-1}x)/\partial x_{j_1}$, $f_l(T^{-1}x)$, whose form is independent of k and such that every monomial occurring in $P(i,j)$ contains at least one factor $\partial g_l/\partial x_{j_1}(T^{-1}x)$ or $f_l(T^{-1}x)$.

Similarly, by differentiating (41) the right number of times and by applying Taylor's formula to the appropriate terms, we have

$$\begin{aligned}&(-\partial^p f_i^{k+1}/\partial x_{l_1} \cdots \partial x_{l_p})(x) \\ &\quad = \sum_{j, m_1, \cdots, m_p} A_{ij}(\partial^p f_j/\partial x_{m_1} \cdots \partial x_{m_p})(T^{-1}x)B_{m_1 l_1} \cdots B_{m_p l_p} \\ &\qquad + P(i, l_1, \cdots, l_p)(x),\end{aligned} \tag{45}$$

where the matrices A and B are given by (43) and (44), and $P(i, l_1, \cdots, l_p)$ is a polynomial in the partial derivatives of n and g of height at most p evaluated at various points in the sphere of radius $\| P_n \circ T^{-1}x \|$ about the point $T^{-1}x$, in the partial derivatives of height at most p of the t^*_j evaluated

at x, in the partial derivatives of *height at most* $p-1$ of the $f^k{}_i$ evaluated at the point $T^{-1}x$.

We now prove Lemma 8 by induction on the height of the partial derivative.

Induction Hypothesis. For $m < l$, the partial derivatives $\partial^m f_i/\partial x_{l_1}\cdots\partial x_{l_m}$ are continuous in B_r, and for every q, there exists a constant A_q such that

$$|(\partial^m f_i/\partial x_{l_1}\cdots\partial x_{l_m})(x)| < A_q \| x \|^q, \text{ for all } (i, l_1, \cdots, l_m). \tag{46}$$

We have proved this for $l=1$; we now assume it for $l=p-1$ and will now prove it for $l=p$.

Now since the g_i (and all its derivatives) vanish at the origin with infinite order, and the various partial derivatives of the t_i and n_i occurring in $P(i, l_1, \cdots, l_p)$ are bounded, for any q (using (46)), we can find a constant $A^*{}_q$ such that

$$|P_i(i, l_1, \cdots, l_p)(x)| < A^*{}_q \| x \|^q. \tag{47}$$

Now we regard $(\partial^p f^k{}_i/\partial x_{l_1}\cdots\partial x_{l_p})(x)$ and $P(i, l_1, \cdots, l_p)(x)$ as the $(i, l_1, \cdots, l_p)$-th rectangular coordinate of vectors $\boldsymbol{f}_p{}^k$ and $\boldsymbol{P}(x)$ in a Euclidean space of m^{p+1} dimensions with the (Euclidean) norm $|||\cdot|||$. Then (45) can be written as

$$\boldsymbol{f}_p{}^{k+1}(x) = \boldsymbol{H}(x)\cdot\boldsymbol{f}_p{}^k(T^{-1}x) + \boldsymbol{P}(x), \tag{48}$$

where $\boldsymbol{H}(x)$ is the linear transformation occurring in (45); it is the tensor product of A and p-times B. Thus, for a point x in $W_{k,r}$,

$$\begin{aligned}\boldsymbol{f}_p{}^k(x) = \boldsymbol{P}(x) &+ \boldsymbol{H}(x)\boldsymbol{P}(T^{-1}x) + \boldsymbol{H}(x)\boldsymbol{H}(T^{-1}x)\boldsymbol{P}(T^{-2}x)\\ &+\cdots+\boldsymbol{H}(x)\cdots\boldsymbol{H}(T^{-k}x)\boldsymbol{f}_p{}^0(x).\end{aligned} \tag{49}$$

In view of the construction of D in $W_{0,r}$ and (47), for any q, there is a constant $A^{**}{}_q$ such that

$$|||\boldsymbol{f}_p{}^0(x)||| < A^{**}{}_q \| x \| \text{ for } x \in W_{0,r} \tag{50}$$

and

$$|||\boldsymbol{P}(x)||| < A^{**}{}_q \| x \| \text{ for } x \in B_r. \tag{51}$$

Now let x^0 be a point of I_+ different from the origin. We wish to show that given any $\epsilon > 0$, there exists a $\delta > 0$ such that for any two points $x^1 \notin I_+$ and $x^2 \notin I_+$, in the sphere of radius δ and about x_0, we have

$$|(\partial^p f_i/\partial x_{l_1}\cdots\partial x_{l_p})(x^1) - (\partial^p f_i/\partial x_{l_1}\cdots\partial x_{l_p})(x^2)| < \epsilon \tag{52}$$

for all $(i, l_1, \cdots, l_p)$.

We can view $\partial^p f/\partial x_{l_1}\cdot\cdot\cdot\partial x_{l_p}$ as the components of the vector $\boldsymbol{f}_p$ as before and replace (52) by

$$|||f_p(x^1)-f_p(x^2)|||<\epsilon. \tag{53}$$

Now the matrix $H(x)$ is uniformly bounded in norm since its entries are just polynomials in $\partial n_i/\partial x_j$ and $\partial t^*_i/\partial x_j$ evaluated at various points in B_r. Thus there is a constant M such that the norm of $H(x)$ is $\leqq M$. Now choose q so large that

$$\kappa^q M=\alpha<1. \tag{54}$$

Applying the same method as was used to prove (37) to (49), we can find a constant K such that

$$\begin{aligned}|||\boldsymbol{f}_p^k(x^1)-\boldsymbol{f}_p^k(x^2)|||\leqq{}&|||\boldsymbol{P}(x^1)-\boldsymbol{P}(x^2)|||\\&+|||\boldsymbol{H}(x^1)\boldsymbol{P}(T^{-1}x^1)-\boldsymbol{H}(x^2)\boldsymbol{P}(T^{-1}x^2)|||\\&+\cdot\cdot\cdot+|||\boldsymbol{H}(x^1)\boldsymbol{H}(T^{-1}x^1)\cdot\cdot\cdot\boldsymbol{H}(T^{-m}x^1)\boldsymbol{P}(T^{m-1}x^1)\\&\qquad-\boldsymbol{H}(x^2)\cdot\cdot\cdot\boldsymbol{H}(T^{-m}x^2)\boldsymbol{P}(T^{m-1}x^2)|||\\&+K\alpha^m\end{aligned} \tag{55}$$

for all large k and all $m<k$. First choose m so large that

$$K\alpha^m<\tfrac{1}{2}\epsilon. \tag{56}$$

Next choose δ_1 so small that if a point $x\notin I_+$ is in the ball of radius δ_1, then $x\in W_{k,r}$ with $k>m$. We now want to make each of the first m terms of the right hand side of (55) less than $\epsilon/2m$. This can be done as follows: By choosing δ sufficiently small, we can arrange that if x^1 is in the ball of the radius δ about x_0, then $T^{-1}x,\cdot\cdot\cdot,T^mx$ is in the ball of radius δ_2 about $T^{-1}x_0,\cdot\cdot\cdot,T^{-m}x_0$ respectively. Now for $y\to I_+$, $F(y)\to 0$ so that for any δ_3, we can choose δ_2 so small that all the derivatives of n_i occurring in the $P(i,l_1,\cdot\cdot\cdot,l_p)T^{-j}x$ $(j=0,\cdot\cdot\cdot,m)$ are evaluated at points in the ball of radius δ_3 about T^jx. Then by the continuity of the partial derivatives of the n_i, t^*_i, g_i, and the partial derivatives of the f_i of height $<p$, we can choose δ_3 so small that each of the first m terms is less than $\epsilon/2m$.

Now an elementary theorem in the theory of real variables (a simple consequence of Taylor's formula) asserts the following: if $u(x_1,\cdot\cdot\cdot,x_n)$ is a function defined and continuous in the neighborhood of a point x_0 lying on a plane I_+: $x_{k+1}=\cdot\cdot\cdot=x_n=0$, such that $\partial u/\partial x_i\leqq k$ exists and is continuous for all i everywhere in the enighborhood of the hyperplane and tends to a limit as $x\to x_0$, then $(\partial u/\partial x_i)(x_0)$ exists and is $\lim_{x\to x_0}(\partial u/\partial x_i)(x)$. Using this

theorem, applied to the partial derivatives of order $p-1$ of f at I_+ ($f=0$ on I_+), we have established that f has continuous derivatives of order p at all points except the origin. But applying the previous argument, we can conclude that for any m, there exists a constant B_m such that

$$|(\partial^p f_i/\partial x_{l_1}\cdot\cdot\cdot\partial x_{l_m})(x)| < B_m |x|^m. \tag{57}$$

Thus the derivatives exist and are continuous (and vanish) at the origin too. We have thus proved

THEOREM 3. *Let $T\in \boldsymbol{G}^n$ and $N\in \boldsymbol{G}^n$ be such that $T-N$ vanishes at the origin with infinite order. Then there exists a $D\in \boldsymbol{G}^n$ such that $DT=ND$ and $D-E$ vanishes at the origin with infinite order. If the invariant surfaces of both N and T are I_+ and I_- and N coincides with T on these surfaces, then D can be chosen to be the identity in a one sided neighborhood of every point of the cone C other than the origin.*

Remark. Theorem 3 implies Theorems 1 and 3 of [14] by the trivial formal considerations of § 4 and of [10] and Lemma 5 of [14].

6. In this section we show that any element T of $\boldsymbol{V}^n$ satisfying (**) lies on a one parameter group. By Theorem 3 and Lemma 4, $T=DND^{-1}$, where $N\in \boldsymbol{C}^n$, $D\in \boldsymbol{G}^n$. Now any element of $\boldsymbol{C}^n$ can clearly be embedded in a one-parameter group of volume preserving transformations. Therefore, there exists a one parameter family T^*_t such that $T^*_1=T$ and such that the Jacobian of T^*_t tends to one with infinite order at the origin. If $v=(v_1,\cdot\cdot\cdot,v_n)$ denotes the vector field corresponding to this flow, then $\operatorname{div} v=\sum \partial v_i/\partial x_i$ vanishes at the origin with infinite order. We now show that we can modify T^*_t so as to become volume preserving. That is we shall prove

THEOREM 4. *Every $T\in \boldsymbol{V}^n$ satisfing (**) lies on a one parameter group $T_s\in \boldsymbol{V}^n$.*

If we consider the vector field v, the equation for a multiplier u so that the field uv be divergence free is

$$\sum \partial(uv_i)/\partial x_i=\sum (\partial u/\partial x_i)v_i+u(\operatorname{div} v)=0 \tag{58}$$

If we write $u(x_1,\cdot\cdot\cdot,x_n)$ as $u(x_1(t),\cdot\cdot\cdot,x_n(t))$, then (58) becomes the ordinary differential equation

$$du/dt=-u(\operatorname{div} v) \tag{59}$$

along each orbit. The general solution of (59) is

$$u(t) = K \exp\left(-\int_0^t \operatorname{div} v(r)\, dr\right).$$

We wish to choose K so that

$$\int_t^{1+t} u(r)\, dr = 1. \tag{60}$$

This can be done if and only if

$$\int_t^{1+t} \operatorname{div} v(r)\, dr = 0. \tag{61}$$

However (61) follows from the fact that T is volume preserving. We thus choose u by (59) and (60) for any point x not on I_+ or I_-. This can be done since $T_t(C)$ sweeps out a whole neighborhood of the origin except for these surfaces. If we set $u = 1$ on $I_+ \cup I_-$, it is easy to see that u is C^∞. Then the one parameter group T_s generated by the vector field uv satisfies the conclusions of the theorem.

Remark. In view of the choice of D, uv agrees with the vector field generating the one parameter group containing N in a one sided neighborhood of every point of C different from the origin.

7. In this section we conclude the proof of Theorem 1.

The cone C is a cross-section to the flow T_s given by Theorem 4 and also to the flow N_s on which N lies. Now, as in § 5, $T_s(C)$, for $\infty < s < +\infty$, sweeps out a whole neighborhood of the origin except for I_+ and I_-. Similarly, $N_s(C)$ sweeps out a whole neighborhood except for I_+ and I_-. Thus, if $y_1, \cdots, y_{n-1}$ denote coordinates on C, we can write, for $x = (x_1, \cdots, x_n) \notin I_+ \cup I_-$,

$$x_i = x_{iT}(y_1, \cdots, y_{n-1}, t). \tag{62}$$

Now

$$\begin{aligned} &(\partial/\partial t)\left(\partial(x_{1T}, \cdots, x_{nT}/\partial(y_1, \cdots, y_{n-1}, t)\right) \\ &\quad = \left(\partial(x_{1T}, \cdots, x_{nT}/\partial(y_1, \cdots, y_{n-1}, t)\right) \qquad (\operatorname{div} uv) = 0. \end{aligned} \tag{63}$$

Thus we can write

$$\partial(x_{1T}, \cdots, x_{nT})/\partial(y_1, \cdots, y_{n-1}, t) = A(y_1, \cdots, y_{n-1}). \tag{64}$$

If we do the same for N, we have, for $x \notin I_+ \cup I_-$,

$$x_i = x_{iN}(y_1, \cdots, y_{n-1}, t). \tag{65}$$

In view of the last remark of § 6, we have

$$\partial(x_{1_N}, \cdots, x_{n_N})/\partial(y_1, \cdots, y_{n-1}, t) = A(y_1, \cdots, y_{n-1}). \tag{66}$$

Thus if we compose the inverse of (62) with (65), we obtain a volume preserving transformation R. We can describe R as follows: if $x = T_s y$ $(y \in C)$, then $Rx = N_s y$. It is clear that R is now defined on an entire neighborhood of the origin and satisfies $RT = NR$. Furthermore, since T and N differ by terms vanishing at the origin with infinite order, R tends to the identity with infinite order at the origin. All that remains to be proved is that R is C^∞ along I_+ and I_-. However, this follows as in § 5.

8. We now consider the group $\boldsymbol{U}^n$ of those local C^∞ mappings which preserve volume up to a constant factor; that is, a map belongs to $\boldsymbol{U}^n$ if and only if its Jacobian determinant is constant. It is clear that the collection of such local maps form a group whose image under the projection onto linear terms is the general linear group $GL(n)$. It follows, as in Section 1, that the subgroups of transformations of the form (i) are maximal commutative subgroups of $\boldsymbol{U}^n$. We now wish to prove

THEOREM 5. *Let $T \in \boldsymbol{U}^n$ have linear part, L, of the form* (i) *and satisfy* (*). *Then there exists an $R \in \boldsymbol{U}^n$ (indeed $\in \boldsymbol{V}^n$) such that*

$$RTR^{-1} = L. \tag{67}$$

We know, by Theorem 1 of [13], that there exists on $R \in \boldsymbol{C}^n$ satisfying (67). The problem is to construct an $R \in \boldsymbol{U}^n$ satisfying (67). We proceed, as usual, in two steps, first proving the corresponding formal theorem.

THEOREM 6. *Let $\mathfrak{U}^n$ be the group of formal power series transformations with constant Jacobian. Then if $T \in \mathfrak{U}^n$ has its linear part L of the form* (i) *and satisfies* (*), *there exists an $R \mathfrak{U}^n$ such that* (67) *holds.*

The proof of Lemma 1 of [9] shows that if we require the linear part of R to be the identity, equation (67) has a unique solution. What we must show is that this solution is indeed volume preserving; i. e., using the notation of (15), that $\Delta(x_1, \cdots, x_n) = \Delta(x) = 1$. Now in view of (67) and the fact that $T \in \mathfrak{U}^n$, the formal power series Δ satisfies

$$\Delta(Lx) = \Delta(x). \tag{68}$$

Comparing coefficients, and noticing that (*) implies that $\lambda_1^{m_1} \cdots \lambda_n^{m_n} \neq 1$ for any positive integers m_i, yields $\Delta \equiv 1$, proving the theorem.

Lemma 3 carries over immediately to the present situation, as do all of the considerations of the previous sections up to the remarks preceding

Theorem 4. That is, after a preliminary change of coordinates belonging to $\boldsymbol{U}^n$, we can assume that $T - L$ vanishes at the origin with infinite order, and that T lies on a one-parameter group of transformations T_t (not necessarily lying in $\boldsymbol{U}^n$). We now prove the analogue of Theorem 4.

THEOREM 7. *Any $T \in \boldsymbol{U}^n$ lies on a one parameter group T_s in $\boldsymbol{U}^n$.*

In case $J(T) = 1$, this follows from Theorem 4. Thus we may assume that $J(T) = a \neq 1$. If v denotes the vector field generating the flow T^*_t, then the equation for a multiplier u so that the vector field uv generates a one parameter subgroup of U^n is

$$\operatorname{div} uv = \alpha = \log a. \tag{69}$$

As in Section 6, this is equivalent to the ordinary differential equation

$$du/dt + u \operatorname{div} v = \alpha \tag{70}$$

along each orbit. The general solution of this equation is

$$u = 1/F(t)\,[\alpha \int_0^t F(s)\,ds + C], \tag{71}$$

where $F(t) = \exp \int_0^t \operatorname{div} v(s)\,ds$. We wish to choose C so that (69) holds. Now $\int_1^{1+x} \operatorname{div} v\,dt = \alpha$ since $T \in U^n$. Therefore, if we divide (70) by u and integrate from x to $1 + x$, we find that (69) can be satisfied if we can choose C in (70) so that

$$\log u(1 + x) - \log u(x) = 0; \tag{72}$$

since $\log F(1 + x) - \log F(x) = \alpha$, we must choose C so that

$$\log[\alpha \int_0^{x+1} F(s)\,ds + C] - \log[\alpha \int_0^x F(s)\,ds + C] = \alpha. \tag{73}$$

Differentiating (72), we obtain

$$(\alpha \int_0^{x+1} F(x)\,ds + C)/F(x+1) = (\alpha \int_0^x F(s)\,ds + C)/F(x),$$

or

$$\alpha \int_0^{x+1} F(s)\,ds + C = \alpha \int_0^x F(s)\,ds + C). \tag{74}$$

Differentiating once more, we obtain an identity. We must thus choose $C = (a - 1)^{-1}\alpha \int_0^1 F(s)\,ds$. Thus, if we take this value for C, we find that

the left hand side of (72) is a constant. If we set $x = 0$, we find that indeed (72) holds. Thus, we have proved Theorem 7.

We can now conclude the proof of Theorem 5 as in Section 7. In fact, the only change is that the differential equation (63) of the form $df/dt = 0$ is replaced by one of the form $df/dt = \alpha f$. This, however, makes no essential difference in the proof.

9. In the following sections, we show how far the techniques of the previous sections apply to the group of all local Hamiltonian maps. A map in the $2n$-dimensional space with coordinates $x_1, \cdots, x_n, y_1, \cdots, y_n$ given by $(x, y) \to (x^*, y^*)$ is said to be Hamiltonian if

$$\sum_{i=1}^{n} dx^*_i \wedge dy^*_i = \sum_{i=1}^{n} dx_i \wedge dy_i. \tag{75}$$

It should be observed that (75) is equivalent to

$$\sum_i [x^*_i, y^*_i]_{x^j x^k} = 0, \tag{76}$$

$$\sum_i [x^*_i, y^*_i]_{x^j y^k} = \delta^k_j, \text{ and} \tag{77}$$

$$\sum_i [x^*_i, y^*_i]_{y^j y^k} = 0, \tag{78}$$

where $[f, g]_{x^j x^k} = (\partial f/\partial x^j)(\partial g/\partial x^k) - (\partial f/\partial x^k)(\partial g/\partial x^j)$ is the usual Lagrange bracket.

We first investigate the group $\mathfrak{H}^n$ of complex Hamiltonian formal power series transformations, i. e. those formal power series transformations in $2n$-space satisfying (76), (77), and (78). The group $\mathfrak{H}^n$ projects onto the symplectic group, $Sp(n)$, under the mapping onto linear terms. A Cartan subgroup of $Sp(n)$ is the set of all matrices of the form

$$\begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & 0 & \\ & & \lambda_n & & & \\ & & & \lambda_1^{-1} & & \\ & 0 & & & \ddots & \\ & & & & & \lambda_n^{-1} \end{pmatrix}. \tag{79}$$

If we apply considerations similar to those of Section 1, we obtain that the formal condition analogous to (*) and (**) is

(***) $\lambda_1^{m_1} \cdot \cdots \cdot \lambda_n^{m_n} = 1$ *for any integers* m_i *(positive or negative)* implies $m_i = 0$.

Furthermore, it is easily seen that the elements of $\mathfrak{G}^n$ commuting with all linear Hamiltonian transformations of thetype (78) have the form

$$x^*_i = x_i f_i(\omega_1, \cdots, \omega_n), \qquad y^*_i = y_i g_i(\omega_1, \cdots, \omega_n), \tag{80}$$

where the f's and g's are formal power series in the products $\omega_i = x_i y_i$. We now need the analogues of Lemmas 1 and 2. If we consider transformation of type (80) and compute the left hand side of (76), we obtain

$$\sum_i [x^*_i, y^*_i]_{x^j x^k} = y_j y_k \sum \omega_i \left((\partial f_i/\partial \omega_j)(\partial g_i/\partial \omega_k) - (\partial f_i/\partial \omega_k)(\partial g_i/\partial \omega_j) \right. \tag{81}$$
$$\left. + y_j y_k (f_j(\partial g_j/\partial \omega_k) - f_k(\partial g_k/\partial \omega_j)) \right).$$

If we also substitute (80) into (77) and (78), we obtain

LEMMA 9. *If* $(x, y) \to (x^*, y^*)$ *is an element of* G^n *of the form* (79), *then*

$\sum_i^n [x^*_i, y^*_i]_{x^j x^k}$ *is of the form* $y_i y_k F(\omega_1, \cdots, \omega_n)$,

$\sum_i^n [x^*_i, y^*_i]_{x_j x_k}$ *is of the form* $y_i y_k G(\omega_1, \cdots, \omega_n)$, and

$\sum_i^n [x^*_i, y^*_i]_{x_j x_k}$ *is of the form* $y_i y_k H(\omega_1, \cdots, \omega_n)$.

In order to determine when a transformation (80) lies in $\mathfrak{H}^n$, we first substitute (80) into (77) for the case $j = k$ and obtain $f_j g_j + \omega_j \partial(f_j g_j)/\partial \omega_j = 1$. The only regular solutions of this equation are $f_j g_j = 1$. Substituting this into (81), we obtain

$$\partial \log f_j/\partial \omega_k - \partial \log f_k/\partial \omega_j = 0. \tag{82}$$

Thus we have proved

LEMMA 10. *The set of all transformations of the form* (80) *belonging to* $\mathfrak{H}^n$ *satisfy* $f_i g_i = 1$ *and* (82). *In particular, they form a maximal commutative subgroup of* $\mathfrak{H}^n$.

We now wish to prove

THEOREM 8. *If* $T \in \mathfrak{H}^n$ *satisfies* (***), *there exists an* $R \in \mathfrak{H}^n$ *such that* $N = R^{-1}TR$ *is of the form* (80),

We set $R = (r_1, \cdots, r_n, s_1, \cdots, s_n)$, where

$$r_i = x_i + \sum r^i{}_{j_1 \cdots j_n, r_1 \cdots r_n} x_1{}^{j_1} \cdots x_n{}^{j_n} y_1{}^{j_1} \cdots y_n{}^{j_n}$$

(83) and

$$s_i = y_i + \sum s^i{}_{j_1 \cdots j_n, r_1 \cdots r_n} x_1{}^{j_1} \cdots x_n{}^{j_n} y_1{}^{j_1} \cdots y_n{}^{j_n},$$

and set $N = (x_2 f_1(\omega_1, \cdots, \omega_n), \cdots, y_1 g_1(\omega_1, \cdots, \omega_n), \cdots)$, where

$$f_i = \lambda_i x_i + x_i \sum f^i{}_{j_1 \cdots j_n} \omega_1{}^{j_1} \cdots \omega_n{}^{j_n}$$

(84) and

$$g_i = \lambda_i{}^{-1} y_i + y_i \sum g^i{}_{j_1 \cdots j_n} \omega_1{}^{j_1} \cdots \omega_n{}^{j_n}.$$

If we substitute (83) and (84) into the equation $RTR^{-1} = N$, we obtain

$$\begin{aligned} r^i{}_{j_1 \cdots j_n, k_1 \cdots k_n} \lambda_1{}^{j_1 - k_1} \cdots \lambda_n{}^{j_n - k_n} + \delta_{j_1}{}^{k_1 + \delta_i{}^1} \cdots \delta_{j_n}{}^{k_n + \delta_i{}^n} f^i{}_{j_1 \cdots j_n} \\ = \lambda_i r^i{}_{j_1 \cdots j_n, k_1 \cdots k_n} + \cdots, \end{aligned} \tag{85}$$

where the $\cdots$ indicate terms of lower height, and a similar set of equations for the s's. Thus if $j_p \neq k_p + \delta_i{}^p$ for some p, then condition (***) implies that $r^i{}_{j_1 \cdots j_n, k_1 \cdots k_n}$ is determined by the terms of lower height. If $j_p = k_p + \delta_i{}^p$ holds for all p, then $r^i{}_{j_1 \cdots j_n, k_1 \cdots k_n}$ is undetermined, whereas the corresponding term in f is. We wish to choose the undetermined terms in R so that $R \in \mathfrak{H}^n$ as we did in Section 2 for $\mathfrak{B}^n$. We choose the undetermined in R so that

$$\begin{aligned} &\sum_{i=1}^{n} [r_i, s_i]_{x_j x_k} \text{ contains no term of the form } y_j y_k \omega_1{}^{l_1} \cdots \omega_n{}^{l_n}, \\ &\sum_{i=1}^{n} [r_i, s_i]_{x_j y_k} - \delta_j{}^k \text{ contains no term of the form } y_j x_k \omega_1{}^{l_1} \cdots \omega_n{}^{l_n}, \\ &\sum_{i=1}^{n} [r_i, s_i]_{y_j y_k} \text{ contains no term of the form } x_j x_k \omega_1{}^{l_1} \cdots \omega_n{}^{l_n}. \end{aligned} \tag{86}$$

We will see later that such choice is possible. We first show that if (86) is satisfied, then R (and hence N) is Hamiltonian. We first remark that if $(x, y) \to (x^*, y^*)$ and $(x^*, y^*) \to (x^{**}, y^{**})$ are two smooth transformations, then

$$\begin{aligned} [x^{**}{}_i, y^{**}{}_i]_{x_j x_k} = &\sum_{l<m} [x^{**}{}_i, y^{**}{}_i]_{x^*{}_l x^*{}_m} (x^*{}_l, y^*{}_l) [x^*{}_l, x^*{}_m]_{x_j x_k} \\ &+ \sum_{l,m} [x^{**}{}_i, y^{**}{}_i]_{x^*{}_l y^*{}_m} (x^*, y^*) [x^*, y^*]_{x_j x_k} \\ &+ \sum_{l<m} [x^{**}, y^{**}]_{y^*{}_l y^*{}_m} (x^*, y^*) [x^*, y^*]_{x_j x_k}, \end{aligned} \tag{87}$$

with similar formulae for $[x^{**}{}_i, y^{**}{}_i]_{x_j y_k}$ and $[x^{**}{}_i, y^{**}{}_i]_{y_j y_k}$.

In particular, if the transformation $(x^*, y^*) \to (x^{**}, y^{**})$ is Hamiltonian, then

$$\sum_i [x^{**}_i, y^{**}_i]_{x_j x_k} = \sum_i [x^*_i, y^*_i]_{x_j x_k}. \tag{88}$$

If we now apply (87) to RN and (88) to TR, we obtain

$$\begin{aligned} &\sum_i \sum_{l<m} [s_i, r_i]_{y_l y_m}(xf, yg)\,[x_l f_l, x_m f_m]_{x_j x_k} \\ &\qquad + \sum_i \sum_{l,m} [s_i, r_i]_{x_l y_m}(xf, yg)\,[x_l f_l, y_m g_m]_{x_j x_k} \\ &\qquad + \sum_i \sum_{l<m} [s_i, r_i]_{y_l y_m}(xf, yg)\,[y_l g_l, y_m g_m]_{x_j x_k} \\ &= \sum_i [r_i, s_i]_{x_j x_k}. \end{aligned} \tag{89}$$

Let us prove

$$\sum_i [r_i, s_i]_{x_j x_k} = 0, \tag{90}$$

for all s, r. It is clear that the linear terms on the left hand side of (90) do vanish. We shall assume that all terms of height smaller than N vanish. If we compare the coefficients of $x_1^{l_1} \cdots x_n^{l_n} y_1^{m_1} \cdots y_n^{m_n}$ in (89), we obtain

$$\begin{aligned} &\Big(\sum_i [r_i, s_i]_{x_j x_k}\Big)_{l_1 \cdots l_n, m_1 \cdots m_n} \\ &\quad = \lambda_j^{-1} \lambda_k^{-1} \lambda_1^{l_1 - m_1} \cdots \lambda_n^{l_n - m_n} \Big(\sum_i [r_i, s_i]_{x_j x_k}\Big)_{l_1 \cdots l_n, m_1 \cdots m_n} + \cdots, \end{aligned} \tag{91}$$

where the $\cdots$ are terms of lower height. Now by Lemma 9, the second term on the right of (91) will vanish unless $l_p = m_p + \delta_j^p + \delta_k^p$. If $l_p \neq m_p + \delta_j^p + \delta_k^p$ for some p, then (91) and (***) imply that

$$\Big(\sum_i [r_i, s_i]_{x_j x_k}\Big)_{l_1 \cdots l_n, m_1 \cdots m_n} = 0. \tag{92}$$

On the other hand, if $l_p = m_p + \delta_j^p + \delta_k^p$ for all p, then the choice (86) implies (91). A similar argument, establishes (77) and (78) for $x^*_i = r_i$, $y^*_i = s_i$.

All that remains to be proved is that we can arrange for (85) to be statisfied. In order to show this, we first observe that in order for a map $(x, y) \to (x^*, y^*)$ to be Hamiltonian and have the identity as Jacobian matrix at the origin, it is necessary and sufficient that there exist a function $H(x, y)$ such that

$$x^*_i = \partial H(x, y^*)/\partial y^*_i \text{ and } y_i = \partial H(x, y^*)/\partial x_i. \tag{93}$$

In fact, (75) is equivalent to

$$\sum x^*_i dy^*_i + \sum y_i dx_i = dH. \tag{94}$$

If we introduce x_i and y^*_i as coordinates, then (93) follows from (94). If

now $(x, y) \to (x^*, y^*)$ is a transformation satisfying (76), (77), and (78) up through terms of order n, then

$$\sum_i x^*_i dy^*_i + \sum_i y_i dx_i = dH + \Omega, \tag{95}$$

where Ω is a linear differential form all of whose coefficients vanish at the origin with order $n+1$. That is, (x^*, y^*) satisfies (92) up through terms of order n. If we replace (x^*, y^*) by exact solutions we shall obtain a Hamiltonian map. Thus, if a mapping $(x, y) \to (x^*, y^*)$ satisfies (76), (77), and (78) up to order n, we can arrange that these equations hold identically by *changing the terms of degree* $> n$ only. Thus if we are given the expansion of R up to order n, we can choose the terms of degree $n+1$ so that R be Hamiltonian up to degree n. However, the only terms of degree $n+1$ in R which can contribute anything to the terms of degree n on the right hand side of (85) are precisely those which are at our disposal. Thus (85) can be satisfied, completing the proof of Theorem 8.

An argument similar to the one given in the preceding paragraph yields the analogue to the present situation of Lemma 3. In trying to apply the techniques of § 5 to the Hamiltonian group, we are faced with a fundamental difficulty. The argument there depends essentially on the fact that no eigenvalue is of absolute value one. This is not, however, implied by (***); we must therefore make the additional assumption. We have thus progressed to the situation analogous to that just before the proof of Theorem 4. The same techniques used to prove Theorem 4 do not apply to the present situation. This is clear, since given a vector field, one can not, in general, find a multiplier which will make it Hamiltonian. However, one can replace the device used in proving Theorem 5 via Theorem 4 by a more pedestrian 'filling in' process applied directly to prove Theorem 5, the details of which we will not bother with here. Modulo these considerations, we have thus proved

THEOREM 9. *Let* $T \in \boldsymbol{H}^n$ *satisfy* (***) *and have no eigenvalue of absolute one. Then there exists an* $R \in \boldsymbol{H}^n$ *such that* RTR^{-1} *is of the form* (80).

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L'APPROXIMATION POLYNOMIALE PONDEREE SUR UN ESPACE LOCALEMENT COMPACT.*

par PAUL MALLIAVIN.

L'approximation polynomiale par Weierstrass de toute fonction continue sur $[0,1]$ a des équivalents classiques pour des intervalles non compacts. Par exemple, l'approximation sur $[0,+\infty)$ par les polynômes de Laguerre multipliés par le poids e^{-x}, ou celle sur $(-\infty,+\infty)$ par les polynômes d'Hermite, le poids étant alors e^{-x^2}. On se propose d'étudier ce problème dans le cadre général dans lequel M. H. Stone [6] a démontré le théorème de Weierstrass.

Le procédé utilisé consistera à se ramener au même problème d'approximation avec une seule variable, problème déjà envisagé par S. Bernstein, S. Mandelbrojt [3], H. Pollard [5]. Ce procédé de réduction donnera en particulier une démonstration du théorème de Stone-Weierstrass n'utilisant pas la structure d'ordre sur l'espace des fonctions continues réelles ([1] et [6]) ce qui permettra d'étendre le résultat de Stone à des algèbres complexes, moyennant des conditions analogues à celles de Mergelyan [4].

1. Notations et énoncés. E étant un espace localement compact, on notera par $C(E)$ l'algèbre des fonctions continues à valeurs complexes définies sur E, par $C_0(E)$ la sous-algèbre des fonctions $f(x)$ tendant vers zéro lorsque x tend vers l'infini; $C_0(E)$ est une algèbre de Banach pour la norme usuelle

$$\|f\| = \max |f(x)|.$$

Si F désigne une partie de $C(E)$, on notera par $\mathcal{A}_F$ la plus petite sous-algèbre contenant F ainsi que l'unité; tout élément de $\mathcal{A}_F$ s'écrit comme un polynôme formé avec un nombre fini d'éléments de F.

Si $F \subset C_0(E)$, le problème d'approximation à partir des éléments de F se ramène au même problème sur un espace compact en compactifiant E par l'adjonction d'un point à l'infini. Ce cas a été envisagé par M. H. Stone dans [6].

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Si $F \not\subset C_0(E)$ alors pour pouvoir approcher des éléments de $C_0(E)$, il est nécessaire d'introduire une fonction $p(x) \in C_0(E)$ telle que $p\mathcal{A}_F \subset C_0(E)$.

On dira que p est *un poids* (pour la famille F) si $p\mathcal{A}_F \subset C_0(E)$ et si $p\mathcal{A}_F$ est dense dans $C_0(E)$.

Deux conditions nécessaires évidentes pour que p puisse être un poids:

$p(x) \neq 0$ pour tout $x \in E$;

F est une famille séparante, c'est à dire que

$f(x) = f(x')$ pour tout $f \in F$ entraine $x = x'$.

Dans toute la suite nous supposerons ces conditions satisfaites.

Remarquons que si p et p_1 sont deux fonctions définies sur E telles que $p = 0(p_1)$, que $p_1\mathcal{A}_F \subset C_0(E)$ et que p soit un poids, alors p_1 est un poids. Sinon on pourrait trouver une mesure $d\mu$ orthogonale à $p_1\mathcal{A}_F$, la mesure $(p/p_1)d\mu = d\rho$ serait orthogonale à $p\mathcal{A}_F$ et satisferait à $\int |d\rho| = 0(\int |d\mu|) < \infty$. p étant un poids, $d\rho = 0$ d'où $d\mu = 0$.

En particulier si E est compact $p = 0(1)$, et $1 = 0(p)$, l'approximation pondérée est équivalente dans ce cas à l'approximation ordinaire.

Enonçons le résultat le plus simple qui permet de reconnaître un poids:

1.1. *Si pour tout $f \in F$, p est à valeurs réelles et*

$$\sum \| pf^n \|^{-1/n} = \infty,$$

alors p est un poids.

Le procédé de démonstration consistera à se ramener au problème à 1 variable suivant. Si e désigne une partie fermée du corps complexe, M_n une suite de nombres positifs, *on dit que la proposition $R(e, M_n)$ est vérifiée* si toute mesure $d\mu$ de support e vérifiant

$$\int z^n d\mu = 0 \text{ pour tout } n \geqq 0, \qquad \int |z|^n |d\mu| \leqq M_n$$

est identiquement nulle.

On dira que $R^*(e, M_n)$ vaut si quel que soit l'entier $s \geqq 0$, $R(e, (M_{2^s n})^{2^{-s}})$ vaut.

On a alors l'énoncé de réduction à 1 variable suivant:

1.2. *Si $R^*(\overline{f(E)}, \| pf^n \|)$ vaut pour tout $f \in F$, alors p est un poids.*

En appliquant au cas compact et en utilisant le théorème de Mergelyan, à savoir: si e une partie compacte non dense du plan complexe, ne séparant pas le plan, alors les polynômes en z sont denses dans l'espace des fonctions

continues sur e, on obtient l'extension suivante du théorème de Mergelyan à un espace compact:

1.3. *Si E est compact, et si pour tout $f \in F$, $f(E)$ est un ensemble non dense, ne séparant pas le plan complexe, alors $\mathcal{A}_F$ est dense dans $C(E)$.*

On peut désirer avoir l'equivalent de 1.3 dans le cas non compact, on obtient l'énoncé suivant contenant 1.1.

1.4. *Supposons que pour tout $f \in F$, $f(E)$ soit un ensemble de mesure superficielle nulle, le complémentaire de $\overline{f(E)}$ se composant de domaines $\Omega_1, \cdots, \Omega_k$, chaque domaine Ω contenant à son intérieur un angle d'ouverture ϕ, et que*

$$\sum \| pf^n \|^{-\pi/\phi n} = \infty,$$

alors p est un poids.

Nous allons commencer par démontrer 1.2.

2. Si la conclusion de 1.2, n'était pas vérifiée, on pourrait trouver une mesure dv portée par E telle que pour tout $f \in F$ et tout n

$$\int pf^n \, dv = 0, \qquad \int |dv| < \infty.$$

Nous allons étudier la projection de cette mesure sur des ensembles de dimension finie.

Si F_1 désigne une partie de F, $\bar{C}$ la sphère de Riemann, c'est-à-dire le plan complexe auquel on a adjoint un point à l'infini, $\bar{C}^{F_1}$ l'espace compact produit de F_1 exemplaires de $\bar{C}$, on a une application continue $h_{F_1}: E \to \bar{C}^{F_1}$ qui à tout $x \in E$ associe $\{f(x)\}_{f \in F_1}$. À la mesure $d\mu$ définie sur E, on associe son image par h_{F_1}, $d\mu_{F_1} = h_{F_1}(d\mu)$, définie par l'égalité

$$\int g(y) \, d\mu_{F_1}(y) = \int g(h_{F_1}(x)) \, d\mu(x)$$

On a alors le lemme:

2.1. *Si pour toute partie finie ϵ de F $d\mu_\epsilon = 0$, alors $d\mu = 0$.*

Démonstration. Soit K une partie compacte de E, g une fonction réelle, continue sur E, de support contenu dans K. La famille F étant séparante, l'application $h_F: E \to \bar{C}^F$ est biunivoque, sa restriction à K est un homéomorphisme et $g \circ h_F^{-1}$ est une fonction continue sur le compact $h_F(K)$. Réalisant un recouvrement de $h_F(K)$ par des ouverts de la forme

$$0 = \{y; y \in h_F(K) \text{ et } |f_i(h_F^{-1}(y)) - a_i| < b_i, i = 1, \cdots, n\}$$

on peut trouver un recouvrement fini de tels ouverts tel que si y et y' appartiennent au même ouvert, on ait

$$|g(h_F^{-1}(y)) - g(h_F^{-1}(y'))| < \eta.$$

Notons par ϵ la partie finie de F constituée par toutes les fonctions f servant à définir les ouverts de ce recouvrement.

On aura en particulier:

$$|g(x) - g(x')| < \eta \text{ pour tout } x \text{ et } x' \text{ vérifiant } h_\epsilon(x) = h_\epsilon(x')$$

Introduisons maintenant la fonction

$$k(x) = \max_{x'} g(x') \text{ où } x' \text{ est assujetti à la condition } h_\epsilon(x') = h_\epsilon(x).$$

On sait que (cf. Bourbaki [2], Chap. V, p. 32) $k(x)$ est une fonction semi-continue supérieurement vérifiant $0 \leqq k(x) - g(x) < \eta$; d'autre part, il existe une fonction l définie sur $h_\epsilon(K)$, semi-continue supérieurement, telle que

$$k(x) = l(h_\epsilon(x)).$$

On aura alors

$$|\int g(x)d\mu(x) - \int k(x)d\mu(x)| \leqq \eta \| \mu \|,$$

$$\int k(x)d\mu(x) = \int l(y)d\mu_\epsilon(y) = 0.$$

η étant arbitraire, on obtient $\int g(x)d\mu(x) = 0$, d'où $d\mu = 0$ ce qu'il fallait démontrer.

Le lemme précédent ramène le problème d'approximation sur un espace localement compact à l'approximation sur les parties de l'espace complexe à m dimensions C^m. Nous allons ramener ce problème au problème à une variable $R^*(e, M_n)$.

2.2. *Soit E une partie de C^m, e_k sa projection sur le $k^{\text{ème}}$ plan complexe de coordonnées, soit $d\mu$ une mesure de support E; posons*

$$\| g \|_\mu = \int |g(z_1, \cdots, z_m)| \, |d\mu(z_1, \cdots, z_m)|.$$

Supposons que

$$R^*(\bar{e}_k, \| z_k^n \|_\mu)$$

soit vérifié, $1 \leqq k \leqq m$, et que $d\mu$ soit orthogonale à tous les polynômes en $z_1, z_2, \cdots, z_m$; alors $d\mu = 0$.

Démonstration. Si $m = 1$, faisant $s = 0$ dans la définition de R^* donnée en 1, on obtient que $R(e_1, \| z_1^n \|_\mu)$ est vérifié, d'où par définition même que $d\mu = 0$. Nous allons établir une récurrence sur la dimension m.

Soit l la projection $C^m \to C^{m-1}$ $\quad (z_1, z_2, \cdots, z_m) \to (z_2, \cdots, z_m)$,

t la projection $C^m \to C$ $\quad (z_1, z_2, \cdots, z_m) \to z_1$,

q étant un entier fixé, projetons par l la mesure $z_1^q\, d\mu$.

$l(z_1^q\, d\mu) = d\rho$ est porté par C^{m-1}, vérifie

$$\int P(z_2, \cdots, z_m)\, d\rho = 0$$

et d'autre part,

$$\| z_k^n \|_\rho < \int | z_k^n | \, | z_1^q | \, | d\mu |, \qquad k = 2, \cdots, m.$$

Appliquons l'inégalité de Schwarz, on obtient

$$\| z_k^n \|_\rho \leqq \| z_k^{2n} \|_\mu^{\frac{1}{2}} \| z_1^{2q} \|_\mu^{\frac{1}{2}}.$$

De cette inégalité il résulte que les propositions $R(\bar{e}_k, \| z_k^{2n} \|_\mu^{\frac{1}{2}})$, $k = 1, 2, \cdots, m$, entrainent les propositions $R(\bar{e}_k, \| z_k^n \|_\rho)$, $k = 2, \cdots, m$, ou en passant à R^*, on obtient que $R^*(\bar{e}_k, \| z_k^n \|_\rho)$ est vérifié. La conclusion de 2.3. était supposée vraie pour $m - 1$, $d\rho = 0$, ce qui peut s'écrire

$$\int k(z_2, z_3, \cdots, z_m) z_1^q\, d\mu = 0$$

quelle que soit la fonction continue k à support compact. Projetons maintenant par t la mesure $k\, d\mu$ sur C, soit $d\sigma = t(k\, d\mu)$. On a

$$\| z_1^n \|_\sigma \leqq \int | z_1 |^n \, | k | \, | d\mu | < \| k \| \, \| z_1^n \|_\mu$$

d'où en appliquant $R^*(\bar{e}_1, \| z_1^n \|_\mu)$ on conclut que $d\sigma = 0$, ce qui s'écrit

$$\int k_1(z_1) k(z_2, \cdots, z_n)\, d\mu = 0$$

quelle que soit la fonction continue à support compact k_1. Les produits $k_1 k$ et leurs combinaisons linéaires étant denses dans $C_0(E)$, on obtient $d\mu = 0$.

2.3. Preuve de 1.2.

Soit dv une mesure orthogonale à $p\mathcal{A}_F$; posons $d\mu = p\, dv$, et si ϵ est une partie finie de F, soit $d\mu_\epsilon$ la projection de $d\mu$ sur $\bar{C}^\epsilon$. Alors on a

$$\| z_i^n \|_{\mu_\epsilon} < \int | f_i^n | \, | p\, dv | < \| p f_i^n \| \int | dv |.$$

Appliquons 2.2 à $d\mu_\epsilon$, on obtient $d\mu_\epsilon = 0$; 2.1 donne $d\mu = 0$, $p(x) \neq 0$ ceci entraine $dv = 0$. c. q. f. d.

3. Nous allons étudier dans un cas simple la proposition $R^*(e, M_n)$.

3.1. *Supposons que e soit mesure superficielle nulle et que chaque com-*

posante connexe du complémentaire de e contienne un angle 0 *d'ouverture* ϕ $(0 < \phi < 2\pi)$, *c'est à dire* $\{z; |\arg(z-z_0)-\alpha| \leqq \frac{1}{2}\phi\}$. *Alors si*

$$\sum M_n^{-\pi/\phi n} = \infty,$$

$R(e, M_n)$ *est vérifié.*

Démonstration. Soit $d\mu$ une mesure vérifiant les hypothèses de $R(e, M_n)$. Posons

$$F(z) = \int d\mu(\xi)/(z-\xi).$$

L'identité

$$1/(z-\xi) = \sum_{p<n} \xi^p/z^p + \xi^n/z^n(z-\xi)$$

donne

$$|F(z)| = |z^{-n}\int \xi^n\, d\mu(\xi)/(z-\xi)| \leqq \|\xi^n\|_\mu\, |z|^{-n}\, d_e^{-1}(z),$$

d_e désignant la distance de z à e. On peut en modifiant le sommet z_0 de angle O se ramener au cas où $d_e^{-1}(z) < 1$, $z \in O$. $F(z)$ sera alors une fonction holomorphe dans O vérifiant

$$|F(z)| < M_n/|z^n| \text{ pour tout } n \geqq 0,$$

d'où $F(z) = 0$, $z \in$ O. (Cf. par exemple [3], p. 27.). Il en résulte que $F(z)$ sera nul sur la composante connexe contenant O, et ainsi $F(z) = 0$ dans le complémentaire de e, c'est à dire $F(z) = 0$ sauf peut-être sur un ensemble de mesure nulle.

Soit $d\lambda$ la mesure ayant une densité superficielle égale à i sur la couronne $r < |z-z_1| < R$. Alors

$$\int d\lambda(\xi)/(-z+\xi) = \psi(z)$$

vérifie $\psi(z) =$ constante $\neq 0$ si $|z-z_1| < r$,

$\psi(z) = 0$ si $|z-z_1| > R$,

$0 \leqq \psi(z) \leqq \psi(z_1)$.

Lorsque l'on fait varier z_1, r, R, les combinaisons linéaires des fonctions ψ sont donc denses dans l'espace des fonctions continues à support compact définies sur le plan complexe.

D'autre part, le théorème de Fubini permet d'écrire

$$\int \psi(z)\, d\mu(z) = -\int F(\xi)\, d\lambda(\xi).$$

F étant nul presque partout, cette dernière intégrale est nulle, d'où $d\mu = 0$.

Remarquons que ce résultat vaut encore si e est contenu sur l'axe réel positif. Il convient alors de prendre $\phi = 2\pi$, comme on le voit en effectuant la représentation conforme $z \to (-z)^{\frac{1}{2}}$.

3.2. Pour passer des relations R aux relations R^* on utilisera les remarques de convexité suivantes:

Soit M_n une suite telle que $\log M_n$ soit convexe, α une constante; alors les séries $\sum M_n^{-\alpha/n}$ et $\sum M_{2n}^{-\alpha/2n}$ sont équiconvergentes. En effet, la seconde série est extraite de la première et d'autre part la convexité implique que pour n assez grand

$$\log M_n < \tfrac{1}{2} \log M_{2n}$$

D'autre part si $\| f^n p \| = M_n$, alors $\log M_n$ est une suite convexe; en effet, si on pose $\psi_x(z) = |f(x)|^z . p(x)$, alors $\log |\psi_x(z)|$ est une fonction harmonique de z, par suite

$$\log \| |f|^z p \| = \sup_x \log |\psi_x(z)|$$

est une fonction sous-harmonique de z, ne dépendant que de $\operatorname{Re} z$, donc c'est une fonction convexe de $\operatorname{Re} z$.

Ces remarques étant faites, la divergence de la série 3.1 implique la relation $R^*(e, \| pf^n \|)$. L'application de 1.2 donne alors la démonstration de 1.4.

Par exemple, si on applique 1.4 au poids e^{-z} on obtient que l'approximation par les polynômes de Laguerre multipliés par e^{-z} est possible sur tout ensemble fermé, ne séparant pas le plan, de mesure superficielle nulle, contenu dans un angle $O = \{z; |\arg z| \leqq \theta < \frac{\pi}{2}\}$.

Remarquons pour terminer que l'on peut préciser davantage la condition 3.1 lorsque e est suffisamment rare. On a par exemple l'énoncé:

Si e est une partie de l'axe réel positif, et si e est réunion d'intervalles $[a_n, b_n]$ où

$$\lim b_n/a_n = +\infty,$$

posons

$$m(x) = \tfrac{1}{2} \int_{E \cap [1,x]} dt/t,$$

$$M(\sigma) = \sup_n (n\sigma - \log M_n).$$

Alors

$$\int_1^{+\infty} M(\log r)\, e^{-\theta m(r)} dr/r = +\infty$$

est une relation $R^*(e, M_n)$ *quelle que soit la constante* $\theta > 1$. *D'autre part, si l'intégrale converge avec* $\theta = 1$, *alors* $R(e, M_n)$ [*et à fortiori* $R^*(e, M_n)$], *n'est pas vérifiée.*

La démonstration de cet énoncé sera donnée dans un autre travail [7].

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ON THE MAXIMALITY OF VANISHING ALGEBRAS.*

By ARTHUR B. SIMON.[1]

Let G be a locally compact Abelian group with invariant Haar measure μ. For any subset S of G, define L_S to be that subset of $L^1(G)$ consisting of all functions whose supports lie in S. When L_S forms an algebra, we call it a vanishing algebra (see [3]). The question naturally arises as to when a vanishing algebra is a maximal subalgebra if $L^1(G)$.

Wermer has proved [4] that if G is a discrete ordered group and S is the set of non-negative elements, then L_S is a maximal proper closed subalgebra of $L^1(G)$ if and only if the ordering in G is Archimedean.

Some time ago, Wermer communicated to me a proof that if G is the real line and S is the right half-line, then L_S is maximal.

In this note, we prove the following converse:

THEOREM. *Let S be a measurable subsemigroup of G. If L_S is a maximal proper closed subalgebra of $L^1(G)$, then G is (continuously isomorphic with) either a discrete subgroup of the reals or the real line itself (with S mapping onto the non-negative part).*

Before this theorem was obtained, it seemed quite likely, at least to the author, that Wermer's second theorem (concerning the real line) could be extended to higher dimensional Euclidean space. Now, however, the same construction used in proving our theorem provides a simple example of a proper closed subalgebra which properly contains all those L^1-functions on the plane whose supports lie in the right half-plane.

It is clear that our theorem coupled with Wermer's results give a complete answer as to when a vanishing algebra is a maximal subalgebra.

Throughout this paper we assume that G is a non-trivial additive locally compact Abelian group with invariant Haar measure μ. We also assume G is a measurable set.

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The space $L^1 = L^1(G)$ of all μ-measurable complex valued functions on G with the property that $\int_G |f(x)|\, d\mu(x) < \infty$ forms a Banach Algebra under the norm $\|f\| = \int_G |f(x)|\, d\mu(x)$ with convolution as multiplication. If f is in L^1 and y is any element of G, then f_y, defined by $f_y(x) = f(x-y)$, is again in L^1. We refer to f_y as the "y-translate" of f.

The essence of our theorem is contained in the following two lemmas:

LEMMA 1. *Let H be a measurable semigroup contained in G such that L_H is a maximal proper closed subalgebra of $L^1(G)$. Then $H \cap -H = \{0\}$.*

Proof. Suppose, to the contrary, that H contains an element $t \neq 0$ such that $-t$ is also in H.

For any subset E of G, we have that $((t+E) \cap H') = t + (E \cap H')$; (here H' denotes the set theoretic complement of H). Hence, if E is measurable, then $\mu((t+E) \cap H') = \mu(E \cap H')$.

Since L_H is proper, H' must be a set of positive measure. Pick e_1 in H' such that every open set about e_1 intersects H' in a set of positive measure; the existence of such a point is guaranteed by [3; Lemma 2.2]. Now $e_1 \neq -t + e_1$, so we can find a pre-compact open set E which contains e_1, and whose closure $\bar{E}$ misses $-t + e_1$. Observe that e_1 is an element of the open set $E \cap (t + \bar{E})'$. Therefore $E_0 = E \cap (t+\bar{E})' \cap H'$ is a set of positive finite measure with the property that E_0, $(t + E_0)$, and H are mutually disjoint.

Set $f_+ = \chi_{(t+E_0)}$ (= the characteristic function of the set $(t + E_0)$) and set $f_- = -\chi_{E_0}$; now let $f = f_+ + f_-$. It follows that f is in L^1, $f \neq 0$, and f is not in L_H. It is also true that if y is any element of G, we have

$$\begin{aligned}\int_{H'} f_y(x)\, d\mu(x) &= \int_{H'} f(x-y)\, d\mu(x) \\ &= \int_{H'} \chi_{(t+E_0)}(x-y)\, d\mu(x) - \int_{H'} \chi_{E_0}(x-y)\, d\mu(x) \\ &= \mu[(t + (y+E_0)) \cap H'] - \mu[(y+E_0) \cap H'] = 0.\end{aligned}$$

Consider the closed linear space I consisting of all those L^1-functions g such that for every y in G, $\int_{H'} g_y(x)\, d\mu(x) = 0$. Since, by definition, $(g_y)_z = g_{y+z}$ always holds, it is clear that if g is in I and y is in G, then g_y is also in I. Thus I is an ideal of $L^1(G)$ (see [2; p. 125]). Now I is a non-trivial ideal and I is not contained in L_H because f is in I.

The set of sums $\{I + L_H\}$ is a subalgebra of $L^1(G)$ which properly contains L_H. It is a proper subalgebra since any element in $L^\infty(G)$ which is constant on H' and identically zero on H will annihilate it. This contradiction shows that $H \cap -H = \{0\}$.

LEMMA 2. *Same hypothesis as Lemma* 1. *Then there is a maximal subsemigroup* F_0 *of* G *such that* $L_{F_0} = L_H$ *and* $F_0 \cap -F_0 = \{0\}$.

Proof. Let $\mathfrak{F} = \{F \subset G: F$ is a measurable semigroup, $H \subset F$, and $L_F = L_H\}$. If F is any element of $\mathfrak{F}$, it follows from Lemma 1 that $F \cap -F = \{0\}$. Let F_0 be the union of a maximal tower in $\mathfrak{F}$. Then F_0 is a semigroup and since F_0' contains a set of positive measure, namely $-H$, it follows that L_{F_0} is proper; $L_{F_0} = L_H$.

We claim that F_0 is closed and therefore measurable. For $H + H \subset F_0$, and thus F_0 has a non-empty interior (the sum of two sets of positive measure has a non-empty interior [1]). Pick x not in F_0; then the set of differences $\{x - F_0\}$ is contained in F_0'. Therefore $\bar{F}_0$ is proper and clearly satisfies the conditions of $\mathfrak{F}$. Therefore $F_0 = \bar{F}_0$.

To show F_0 is maximal, let R be any semigroup containing F_0. If $R \neq G$, pick x in R'. Then, as above, $\bar{R}$ is proper and $L_R = L_H$. By the maximality of F_0 in $\mathfrak{F}$, we have $R = F_0$, and Lemma 2 is proved.

Proof of the Theorem. In view of Lemma 2, there is no loss of generality in assuming that S is a maximal semigroup and $S \cap -S = \{0\}$. It follows that $G = S \cup -S$ (see [1; Lemma 4.1]). Let S^0 be the interior of S and S_1 the open semigroup $S^0 - \{0\}$. Let T be any open semigroup which properly contains S_1. If $S' \cap T = \phi$, then $T \subset S^0$, so T must contain 0. If $S' \cap T \neq \phi$, then $-S^0 \cap T \neq \phi$ and 0 is again in T. Thus S_1 is a maximal 0-proper open semigroup, using the terminology of F. B. Wright [5]. Continuing in this vein, the semigroup $s(S_1) = \{x \in G: x + S_1 \subset S_1\}$ must be equal to S; for clearly $S \subset s(S_1)$, and since 0 is not in S_1, no element of $-S_1$ is in $s(S_1)$; equality follows because S is maximal. For the same reasons, $s(-S_1) = -S$. Since $S \cap -S = \{0\}$, we know that G is what Wright calls a Hölder group. In the aforementioned paper, it is proved that there is a continuous isomorphism from a Hölder group to the group of reals.

According to the structure theorem for locally compact Abelian groups, we know $G = R^n \oplus G_1$ with $G_1/K = D$, where K is compact, R^n is Euclidean n-space, and D is discrete. The subgroup K must reduce to a point since there are no compact subgroups of the reals. Therefore $G = R^n \oplus D$. If $n \neq 0$, then the image of R^n is connected; hence all of the real line and D

must be just a point. If $n = 0$, then G is discrete and we may also consider the image of G under the isomorphism to be discrete. This completes the proof.

Remarks. (1) We could have assumed that S was a closed subset of G rather than a semigroup since in that case we have proved in [3] that S is the union of a semigroup and a set of measure zero.

(2) In the proof of the theorem, we eliminated the compact group K; i. e. we showed that G contains no non-trivial compact subgroup. We can also prove that in the R^n component, n must be 0 or 1. For, if $n \neq 0$, then D must reduce to a point. Now we have shown that S can be taken to be a maximal, hence closed, semigroup with the property that $S \cap -S = \{0\}$ and $G = S \cup -S$. Clearly, only R^1 contains a subset of this nature.

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LINE ELEMENTS ON THE TORUS.*

By BRUCE L. REINHART.[1]

The aim of this paper is to study the relations between the integral curves of a (oriented or nonoriented) line element field on the torus and the homotopy invariants defined by considering it as a map from the torus into the circle. The principal tools are generalizations of the winding number, rotation number, and cycle without contact defined originally by Poincaré [4, pp. 25, 73, 145], and the elementary properties of the first homology group of the torus. Let $F: T^2 \to S^1$ be the line element field and F_* the induced map on the first homology group. Then any closed integral curve defines an element of the kernel of F_*. If $F_* \neq 0$, there exists a closed integral curve, the homology class of which generates the kernel. F_* is characterized by a pair of integers (i, j). In case $F_* \neq 0$, the rotation number may be expressed in terms of i and j; in particular, it is always a rational number. A new proof of the classification theorem of Kneser [3] is given, and applied to establish necessary and sufficient conidtions for orientability of a nonoriented line element field, and the best possible homotopically invariant lower bound for the number of closed integral curves. Finally, a few remarks are made about line element fields on the Klein bottle.

1. The winding number on the torus. For the purposes of this paper, we shall consider the torus as the quotient of the (x, y) plane by the lattice of points with integer coordinates. This representation induces in a natural way a parallelization of the torus, which we shall take as fixed throughout. We shall permit ourselves to make changes of coordinates in the plane by integer matrices of determinant one and by translations; such a change induces a one-one map of the torus onto itself which takes parallel vectors into parallel vectors.

We shall concern ourselves with the study of line element fields on the torus; by this we mean C^1 cross sections into the tangent sphere bundle (oriented line elements) or projective space bundle (nonoriented line elements).

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Using the canonical parallelization, such a cross section may be considered as a map F of the torus T^2 into the sphere S^1 or the projective space P^1. In classical terminology, F is just the Gauss spherical map. A line element field may be represented by a (oriented or nonoriented) vector field, in the sense that at each point the vector is in the direction of the line element assigned; for example, the field of unit vectors with this property. Conversely, any vector field gives rise to a line element field by ignoring the length of the vector and considering only the direction. By this means, we may define the notion of a singular line element field: beginning with a C^1 vector field with isolated zeroes, we construct a line element field defined everywhere except on a set of isolated points.

Let F be a continuous map of T^2 into S^1 and let $F_*: H_1(T^2) \to H_1(S^1)$ be the induced homology map. By elementary obstruction theory [6, § 34.6], the homotopy classes of such maps F are in one-one correspondence with the elements of $H^1(T^2; \pi_1(S^1))$, that is, with the maps F_*. The latter maps are characterized by a pair (i, j) of integers such that $F_*(z_1) = iz'$ and $F_*(z_2) = jz'$ where $\{z_1, z_2\}$ is a basis for $H_1(T^2)$ and $\{z'\}$ is a basis for $H_1(S^1)$. By the differentiable approximation theorems [6, § 6.7], the same holds for C^1 homotopy classes of C^1 mappings.

In order to study the mapping F_*, it is convenient to consider the restriction of F to a closed curve C lying on T^2. By considerations similar to those of the preceding paragraph, it is seen that the homotopy classes of maps of a circle into a circle are determined by a single integer k, called the degree of the map. We may define the winding number $I_F(C)$ of the closed curve C with respect to the field F as the degree of the restriction of F to C. It is independent of admissible coordinate changes in the plane, since the induced map on tangent directions is one-one and orientation preserving, hence is of degree 1. The notion of winding number has been used by numerous authors in studying singularities of a vector field. We shall agree to parametrize S^1 by $\frac{1}{2}\pi$ times the angle between a given vector and the positive x axis, and P^1 by $1/\pi$ times this angle. Then the following proposition relates the various usual definitions of winding number and gives a means of computation. The proposition deals with regular simple closed curves; these are simple closed curves which have everywhere a continuous nonzero tangent vector. Such curves are differentiable submanifolds of class C^1.

PROPOSITION 1. *Let F map the regular simple closed curve C into S^1. Let θ be a parametrization of S^1 by the real numbers modulo 1 and let x_0 be a point of S^1 such that $F^{-1}(x_0)$ is a finite set of points. Let p be the*

number of times that the value x_0 is taken on as θ increases, and n the number of times as θ decreases. Then $k = \int_C F^* d\theta = p - n$.

Proof. The fundamental cycle of C is mapped into k times the fundamental cycle of S^1; hence, the fundamental cocycle of S^1 is mapped into k times the fundamental cocycle of C. But $\int_{S^1} d\theta$ is a representation of the fundamental cocycle of S^1, so $\int_C F^* d\theta$ represents the induced cocycle on C. Hence the latter integral is equal to k as claimed. The points of $F^{-1}(x_0)$ may be divided into three categories according as θ increases, decreases, or attains a local extremum. Let $q_1, \cdots, q_t$ be the set of points of the first two categories, arranged in their natural order on C. Then

$$k = \int_C F^* d\theta = \theta \begin{vmatrix} F(q_2) \\ F(q_1) \end{vmatrix} + \cdots + \theta \begin{vmatrix} F(q_t) \\ F(q_{t-1}) \end{vmatrix} + \theta \begin{vmatrix} F(q_1) \\ F(q_t) \end{vmatrix}$$

where θ is to be taken as a real number (not modulo 1). If the two limits of any term are in the first category, the integral is $+1$; if in the second category, -1; and if in different categories, 0. Let p^+ be the number of points in the first category which are followed by a point of the same category, p^- the number of those followed by a point of the second category, n^+ the number of points of the second category which are preceded by a point of the first category, and n^- the number which are preceded by a point of the second category. Then $k = p^+ - n^- = p^+ + (p^- - n^+) - n^- = p - n$.

THEOREM 1. *The winding number of a regular simple closed curve C on T^2 with respect to its oriented tangent vector is ± 1 or zero according as C bounds or not. For the nonoriented case, the numbers are ± 2 or zero respectively.*

Proof. Lift C to a simple arc C' in the plane covering C once. C' is a C^1 curve with nonzero tangent vector, such that the tangents at the two end points are parallel. C' is a closed curve if and only if C bounds; in this case, the winding number of C' in the plane, which clearly equals the winding number on the torus, is well know to be ± 1 [1]. If C does not bound, we construct a regular curve in the plane as follows: Let P' and Q' be the endpoints of C', and let l be parallel to $P'Q'$, tangent to C', and such that C' lies entirely on one side of l. Let l meet C' in P, and let PQ be an arc covering C once. Let RS be a line parallel to l and separated from PQ by l. Join PQ to RS by a pair of semicircles tangent to each, thus making a closed curve

$PQSR$. Now the winding number of $PQSR$ with respect to its tangent vector is 1 (respectively -1) while the semicircles each contribute $\frac{1}{2}$ (respectively $-\frac{1}{2}$) to the integral and the line SR contributes nothing. Hence PQ contributes zero; but the integral giving its contribution to the winding number of $PQSR$ clearly gives the winding number of C. Since the canonical map of S^1 onto P^1 is of degree 2, all winding numbers are doubled in that case.

COROLLARY 1. *The homology class defined by a nonbounding closed integral curve of a line element field F lies in the kernel of the homology map F_*. In particular, if F is nonsingular and has a closed integral curve, these conditions are satisfied.*

Proof. The first statement is immediate. If F is nonsingular, no closed integral curve can bound, for the index of such a curve is equal to the total of the indices of the singularities enclosed [1].

COROLLARY 2. *Let C be a nonbounding regular simple closed curve which is tangent to F at a finite number of points. Then $I_F(C) = P - N$, where P is the number of points where F is tangent as the angle increases and N is the number as the angle decreases.*

Proof. Since the winding number of C with respect to its tangent vector is zero, we may deform the map $G_1: C \to S^1$ which takes each point of C into the tangent vector at that point into the mapping $G_2: C \to 0$. Let $\theta(t)$ be the angle between $(F \mid C)(t)$ and the tangent vector to C at $C(t)$ for some parametrization of C. Then $(F \mid C)(t) = G_1(t) + \theta(t)$. Using the homotopy between G_1 and G_2, we see that $(F \mid C)$ is homotopic to the map $F': C(t) \to \theta(t)$. This map takes on the value 0 only at points of tangency of F with C. Using Proposition 1, Corollary 2 follows immediately.

For the remainder of the paper we shall assume that F is nonsingular. Our immediate goal is to apply our knowledge of $F \mid C$ to derive certain facts about F. Let us first recall a few elementary properties of the first homology group of the torus.

LEMMA 1. *Let $\{\alpha, \beta\}$ form a basis for $H_1(T^2)$ and let γ be a nonbounding simple closed curve. Then γ is homologous to $m\alpha + n\beta$ with m and n relatively prime; or equivalently, γ may be taken as an element of a new basis $\{\gamma, \gamma'\}$. Moreover, if a connected curve $\tilde{\gamma}$ in the plane covering γ passes through (x_0, y_0), the nearest points on $\tilde{\gamma}$ of the form $(x_0 + i, y_0 + j)$ with i and j integers are $(x_0 + \epsilon m, y_0 + \epsilon n)$, where $\epsilon = \pm 1$. Here we have assumed that α (respectively β) is homologous to the projection of the x (respectively y) axis.*

LEMMA 2. *Let α and α' be simple closed curves on T^2. If they are nonintersecting and nonbounding, then they are homologous. If their intersection number is nonzero, they neither bound nor are homologous.*

Proof. Choose a basis $\{\alpha, \beta\}$ for $H_1(T^2)$; then the intersection number $\alpha \cap \beta = 1$, and $\alpha' \sim m\alpha + n\beta$ with m and n relatively prime. But $\alpha \cap \alpha' = \alpha \cap (m\alpha + n\beta) = n(\alpha \cap \beta) = n$, so $n = 0$ and $m = 1$ as required. The second part follows immediately from the facts that $\gamma \cap \gamma = 0$ and $\gamma \cap \gamma' = 0$ whenever γ' bounds.

THEOREM 2. *If there exists a nonbounding simple closed curve Γ not homologous to a closed integral curve and such that $I_F(\Gamma) = 0$, then $F_* \equiv 0$. In particular, if F has no closed integral curves, these conditions hold.*

Proof. (i) Suppose first F has a closed integral curve C. Since by assumption Γ and C are not homologous, neither is homologous to zero, and each is a simple closed curve contained in the kernel of F_*, it follows by Lemma 1 and elementary properties of the integers that $F_* \equiv 0$.

(ii) Suppose there does not exist a closed integral curve. Then there exists a regular simple closed curve Γ' never tangent to F, such that every semiorbit of F meets Γ' eventually and always crosses Γ' in the same direction [1, p. 416; 5]. Here by semiorbit we mean the curve formed by beginning at an arbitrary point P and following the integral curve through P in one direction. Let P be a point of Γ', and consider a semiorbit through P. This semiorbit must meet Γ' infinitely often. Let Q be a point of accumulation of these intersections and N a flat neighborhood of Q, that is, a neighborhood with coordinates (x, y), $|x| \leqq 1$, $|y| \leqq 1$, such that the integral curves of F in N coincide with the level lines of y. These level lines will be called plaques of N. Since Γ' is never tangent, we may assume N small enough to that the angle between the tangent to Γ' at q and $F(p)$ is greater than ϵ for all p in N and q in $\Gamma' \cap N$. Let N' be a flat subneighborhood of N of such shape that any two plaques in it may be joined into a regular curve which makes an angle less than $\epsilon/2$ with any plaque, and small enough so that $\Gamma_1 = \Gamma' \cap N'$ is connected. Pick Q_1 in Γ_1 and lying on the semiorbit through P. Follow the semiorbit, labeling the successive intersections with Γ' $Q_2, \cdots, Q_{k+1}$, where Q_{k+1} is the next one lying on Γ_1. Q_1 and Q_{k+1} are crossings in the same direction; hence we may join the plaques through them in such a manner as to construct a regular simple closed curve which differs from an integral curve only in N' and which crosses Γ' in the same direction. Call this curve C. The angle between C and F may be

assumed to be less than $\epsilon/2$. By Corollary 2, Γ defines an element of the kernel of F_* because it is never tangent. C defines an element of the kernel because $F \mid C$ differs from the map of C into its tangent vectors by less than $\epsilon/2$, so the maps are deformable into each other. Since C and F cross always in the same direction, the intersection number is nonzero; hence by Lemma 2 neither bounds, and C and F are not homologus. We may then proceed as in case (i) to show that $F_* \equiv 0$. It follows in particular that $I_F(\Gamma) = 0$ for any Γ satisfying the hypotheses of the theorem. This completes the proof.

COROLLARY 3. *If $F_* \not\equiv 0$, F has a closed integral curve.*

Criterion. Suppose there exists a closed curve C such that $I_F(C) \neq 0$. Then F has a closed integral curve. If C' is a nonbounding closed curve not homologous to C and $I_F(C')$ is known, then the homology class of the closed integral curve may be determined by finding the kernel of F_*.

2. Cycles of minimal contact. If F is a fixed line element field and C and C' are two curves which are homotopic by a homotopy which that does not pass through any singular points of F, then $I_F(C) = I_F(C')$. Indeed, the given homotopy induces a homotopy between $F \mid C$ and $F \mid C'$, so the degree of the two maps must be the same. In particular, if F is nonsingular, the winding number $I_F(C)$ depends only upon the homology class of C. Let us set the following problem: Given a homology class on T^2, find a curve in it which is tangent to F at the fewest points. Such a curve will provide the most direct computation of the winding number, by Corollary 2. Throughout the remainder of the paper it will be assumed that F is nonsingular.

Let us fix a homology class which contains a simple closed curve, but does not contain a closed integral curve. This is possible because any two closed integral curves are nonintersecting simple closed curves, so are homologous by Lemma 2.

LEMMA 3. *Let C be a simple closed curve not homologous to a closed integral curve of F or to zero. Then C is homologous to a regular simple closed curve C^* such that:*

(i) *C^* is tangent to F at only a finite number of points.*

(ii) *Every semiorbit of F meets C^* eventually.*

Proof. Recall that F has been assumed nonsingular. Hence at each point we may find a flat coordinate neighborhood. Replace C by a regular curve, also called C, which is homologous to it. Cover C by a finite number

of flat coordinate neighborhoods U_α such that the closure $\bar{U}_\alpha$ is a concentric subsquare in V_α which is also flat. Since C is locally connected, we may at each point of $C \cap \bar{U}_\alpha$ find a neighborhood $W_\beta \subset V_\alpha$ whose intersection with C is connected. Each component of $C \cap \bar{U}_\alpha$ must be covered by a finite number of these, as must the whole set $C \cap \bar{U}_\alpha$; hence this set has finitely many components C_γ. We may cover C by a finite number of the interiors of the sets C_γ; call these sets D_δ and number them consecutively around C from some arbitrary starting point, discarding unnecessary ones.

The lemma may now be proved by induction on δ as follows: D_1 is contained in some $\bar{U}_\alpha$ which contains finitely many of the C_γ. Each C_γ is a connected arc beginning and ending on the boundary of $\bar{U}_\alpha$. Let 2ρ be less than the minimum distance from $\bar{D}_1$ to the other C_γ in $\bar{U}_\alpha$, and describe about D_1 the tube of radius ρ. We shall construct a regular arc lying in the tube and in $\bar{U}_\alpha$, having the same initial and final points and tangent vectors as $\bar{D}_1$, and being tangent to a plaque at only a finite number of points. By the rectifiability of C, we can approximate $\bar{D}_1$ by a finite number of line segments of length less than ρ with endpoints on $\bar{D}_1$; such segments are surely contained in the tube and in $\bar{U}_\alpha$, so do not meet the other C_γ. If any of them lies in a plaque, we may turn it slightly; then $\bar{D}_1$ is approximated by a polygonal arc never lying in a plaque. Finally, we may at each corner smooth out to a regular curve by putting in the arc of a circle. In this way, we get at most one point in each corner where the curve is tangent to a plaque. Special care must be exercised at the endpoints of $\bar{D}_1$; but the same effect may be achieved. In applying this construction to D_2, we actually apply it to that subset of D_2 beginning at the endpoint of $\bar{D}_1$. Proceeding in this manner, statement (i) is proved.

To prove (ii), we consider two cases according as F does or does not have closed integral curves. If it has closed integral curves, they by assumption are not homologous to C^*, hence have nonzero intersection number with it, so must cross it. Any other semiorbit is contained in an annulus formed by cutting the torus along some integral curve. Hence, it must spiral toward some closed integral curve from one side, by the Poincaré-Bendixson theorem [1]. Consequently it also must cross C^*. If F has no closed integral curve, cut T^2 along C^*. If any semiorbit stayed in the annulus thus formed, it would have to spiral toward a closed integral curve; that being excluded, it must meet C^* eventually. This proves Lemma 3.

Lemma 3 justifies the following definition.

Definition. A cycle of minimal contact Γ is a regular simple closed curve,

lying in a given homology class not containing a closed integral curve of F, such that the number of points of tangency is as small as possible. We shall indicate by $\mu(\Gamma)$ the winding number of Γ.

PROPOSITION 2. *Let Γ be a cycle of minimal contact such that $\mu(\Gamma) = 0$. Then $F_* \equiv 0$.*

Proof. This follows immediately from Theorem 2.

3. The rotation number. In the classical theory of nonsingular differential equations on the torus, a cycle without contact was much used. We shall use instead a cycle of minimal contact; throughout this section, Γ will denote such a cycle. By use of Γ we shall generalize the classical invariant called the roation number; this invariant will be denoted by $\lambda(\Gamma)$. We shall also find the relation between $\mu(\Gamma)$ and $\lambda(\Gamma)$ on one hand and the integers which characterize F_* on the other. In defining $\lambda(\Gamma)$, we shall assume there exists a closed integral curve, since the contrary case can be handled by known methods [4, 5]. Let us assume we have chosen coordinates in the (x, y) plane so that Γ is homologous to the cycle covered by $x = 0$ and passes through the projection of $(0, 0)$. Then the inverse image of Γ in the plane contains all the integer points, and each component contains all such points with a fixed x, by Lemma 1. Parametrize Γ differentiably by t, $0 \leqq t \leqq 1$; this induces a parametrization by t, $-\infty < t < \infty$, on all the curves $\tilde{\Gamma}_i$, $i = 0, \pm 1, \cdots$, passing through $(i, 0)$ and covering Γ, if we understand that $t = 0$ at the points $(i, 0)$. Let Γ' be the curve covered by $y = 0$.

LEMMA 4. *If a semiorbit C' either* (i) *crosses in opposite directions at sucessive points of crossing A and B, or* (ii) *crosses at A and is tangent at B on the side to which it crosses, or* (iii) *is tangent at A and crosses at B from the side on which it is tangent, then the arc AB on C' together with a suitably chosen arc BA on Γ bounds, and contains a point of tangency of Γ with F.*

Proof. Consider the semiorbit from A toward B, and lift it up into one of its connected inverse images $\tilde{C}$ in the plane. We may suppose that A lies on that inverse image $\tilde{\Gamma}$ of Γ which contains $(0, 0)$. Now $\tilde{\Gamma}$ and each of its translates separate the plane into two components; hence, if B lay on another inverse image of Γ, C would cross at B in the same direction as at A. Hence, B must also lie on $\tilde{\Gamma}$. The arc AB on $\tilde{\Gamma}$ is well defined, and its projection is the desired arc on Γ. It remains to find a point of tangency. Suppose there is none. Let C' be the closed curve: AB on $\tilde{C}$ followed by BA on $\tilde{\Gamma}$. Let P be the midpoint of AB and consider the semiorbit through P

entering the region bounded by C'. If this semiorbit were to remain inside, there would be a closed integral curve, which would have to bound and hence to surround singular points. Thus the semiorbit leaves at Q. Call whichever of P and Q is closest to A, A_1 and the other, B_1. The curve: A_1B_1 along the semiorbit followed by B_1A_1 along $\tilde{\Gamma}$ will be called C_1'. Continuing in this manner, we construct a series of closed intervals A_iB_i whose intersection is a single point P. P is a point of tangency, for if not there would be a semiorbit entering the interior of C' at P which could not leave again, since its point of leaving must lie on all the arcs A_iB_i of $\tilde{\Gamma}$. Since that is impossible, the lemma is proved.

LEMMA 5. *A semiorbit C of an open integral curve, lifted to $\tilde{C}$ in the plane, crosses the $\tilde{\Gamma}_i$ in the same direction and in monotone order, with possibly a finite number of exceptional crossings. A semiorbit of a closed integral curve C' which is homologous to $m\Gamma + n\Gamma'$ has the property that each covering $\tilde{C}'$ first crosses the curves $\tilde{\Gamma}_{kn}$, $k = \pm 1, \pm 2, \cdots$, in monotone sequence and in the same direction. Similar results hold for the curves $\tilde{\Gamma}_{kn+\alpha}$, with $|kn + \alpha| \geqq |n|$.*

Proof. Since there are a finite number of points of tangency between Γ and F, there are a finite number on C. Let us go far enough on C to be beyond these. If C crosses $\tilde{\Gamma}_i$ twice, it must do so in opposite directions; hence C crosses Γ twice in opposite directions at successive points and marks off an interval D_1 on Γ in which there is a point of tangency. If C reenters the region E_1 bounded by D_1 and a portion of C, there is delineated a subinterval D_{11} on which there is a point of tangency, and C must eventually leave by crossing $D_1 - D_{11}$. If C reenters E_1, it must do so through $D_1 - D_{11}$ and mark off another interval D_{12} on which there is a point of tangency. Since there are a finite number of points of tangency on Γ, C must eventually leave the region E_1 permanently. In a similar manner, a finite number of arcs D_i and regions E_i are defined; when C has passed all these, $\tilde{C}$ must cross the $\tilde{\Gamma}_i$ in monotone sequence as claimed. Now let C' be a closed integral curve, and let $\tilde{C}'$ cover it. Consider a semiorbit starting at (x_0, y_0) on Γ_0. C' passes through $(x_0 + n, y_0 + m)$, but it may meet $\tilde{\Gamma}_n$ first at (x_1, y_1). By periodicity, the first meeting with $\tilde{\Gamma}_{kn}$ will be at $(x_1 + kn, y_1 + km)$. These points are in order on $\tilde{C}'$ because each $\tilde{\Gamma}_{kn}$ separates the plane. The remainder of the lemma is proved similarly.

We shall call a semiorbit positive if eventually the sequence described in Lemma 5 is monotone increasing, and negative if it is monotone decreasing. Note that a given orbit may have two semiorbits of the same sign.

Let P be a point of $\tilde{\Gamma}_0$ and arbitrarily choose one of the semiorbits through it. If this projects into an open semiorbit, choose $i_0 \neq 0$ large enough in absolute value so that the $\tilde{\Gamma}_i$ are crossed in monotone sequence for $|i| > |i_0|$, and set $\phi_i(P)$ to be the t coordinate of the crossing with $\tilde{\Gamma}_i$ for such i. If the semiorbit projects into a closed orbit, put $\phi_i(P)$ equal to the t coordinate of its first crossing with $\tilde{\Gamma}_i$, $i \neq 0$. Then define

$$\lambda'(\Gamma, P) = \lim_{i \to \infty} \phi_i(P)/i.$$

Similarly define $\lambda''(\Gamma, P)$ for the other semiorbit.

LEMMA 6. *$\lambda'(\Gamma, P)$ and $\lambda''(\Gamma, P)$ are equal rational numbers which depend only upon Γ. Moreover, if the closed integral curves belong to the class $m\Gamma + n\Gamma'$, then $\lambda = m/n$.*

Proof. Consider first a closed orbit C, homologous to $m\Gamma + n\Gamma'$. According to our parametrization of $\tilde{\Gamma}_i$, if (x_1, y_1) has parameter t, then $(x_1 + kn, y_1 + km)$ has parameter $t_1 + km$. Hence

$$\lim_{k \to \infty} \phi_{kn}(P)/kn = m/n$$

and similarly for the terms $kn + \alpha$. Hence $\lambda' = \lambda'' = m/n$, and is thus independent of the choice of closed integral curve. But any open orbit is asymptotic to a closed orbit, from which it readily follows that the limit in that case is the same.

Remark. Since m and n are relatively prime, λ and the pair (m, n) determine each other.

Fix the basis $\{\Gamma, \Gamma'\}$ for $H_1(T^2)$ as above. Then the homotopy class of F is defined by the pair of integers (i, j), namely the winding numbers of these basis elements (see § 1).

THEOREM 3. *$\mu(\Gamma) = i$. If $\mu(\Gamma) = 0$, then also $j = 0$. If $\mu(\Gamma) \neq 0$, then $\lambda(\Gamma) = -j/i$.*

Proof. The first statement is obvious, and the second follows from Proposition 2. If $\mu(\Gamma)$ is not zero, there must exist a closed integral curve; let its homology class be $m\Gamma + n\Gamma'$ with m and n relatively prime. We know that this class belongs to the kernel of F_*. Its image under F_* is $mi + nj = 0$, from which we see that $m/n = -j/i$. But by Lemma 6, $\lambda(\Gamma) = m/n$.

4. Classification theorem. In this section we shall apply the cycle of minimal contact to describe qualitatively the integral curves of a nonoriented

line element field. This gives a new proof of a theorem of H. Kneser [3], proved by him by methods of combinatorial topology. We shall deal primarily with the case where there exists a closed integral curve, since the contrary case has been suitably handled elsewhere [2, 5, 7]. By using this classification, we shall prove simple necessary and sufficient conditions for orientability, and give the best possible homotopically invariant lower bound for the number of closed integral curves in terms of homotopy invariants of the field F.

Let us therefore suppose there exists a closed integral curve C, which by change of coordinates may be taken to be homologous to the curve covered by $y = 0$. Let us choose a cycle Γ of minimal contact which is homologous to the cycle covered by $x = 0$. We may suppose that C and Γ meet on the image of $(0, 0)$. Let $\tilde{C}_i$ be that component of the inverse image of C which passes through $(0, i)$ and $\tilde{\Gamma}_i$ similarly for Γ and $(i, 0)$. Then the region bounded by $\tilde{\Gamma}_0$, $\tilde{\Gamma}_1$, $\tilde{C}_0$, and $\tilde{C}_i$ is a fundamental domain, in the sense that it maps onto the torus, and the map is one-one at interior points. Moreover, if C' is an integral curve different from C, there is a component of its inverse image which lies between C_0 and C_1. Let us agree to abbreviate $\mu(\Gamma)$ by μ throughout this section.

LEMMA 7. *Let C' be a semiorbit of an integral curve of F. Then none of the following can occur:*

(i) *C' is tangent to Γ and crosses at the point of tangency.*

(ii) *C' crosses Γ in alternating directions at three crossings in order on C' (not necessarily successive).*

(iii) *C' is tangent to Γ at two points.*

(iv) *C' is tangent from one side and at another crossing goes from that side to the other.*

Proof. (i) If this situation occurs, consider a flat neighborhood N containing just one point of tangency. In N we have a family of plaques, with $\Gamma \cap N$ tangent to one of them at one point. Clearly Γ could be slightly altered in N so as not to be tangent, contradicting its minimal property.

(ii) Let the semiorbit $\tilde{C}'$ in the plane cover C', and suppose for definiteness that the first time $\tilde{C}'$ crosses one of the $\tilde{\Gamma}_i$, it does so from left to right. Let Q be the first crossing from right to left, and $\tilde{P}$ the preceding crossing; both these lie on a particular $\tilde{\Gamma}_i$. If $\tilde{C}'$ again crosses $\tilde{\Gamma}_i$, it mus do so from left to right; let $\tilde{R}$ be the first point where this occurs. If $\tilde{C}'$ does not again cross $\tilde{\Gamma}_i$, it must be a negative semiorbit. In that case, we choose $\tilde{Q}'$ to be

the first point beyond $\tilde{Q}$ where $\tilde{C}'$ crosses from left to right, and $\tilde{P}'$ the next preceding crossing, both lying on $\tilde{\Gamma}_j$. The semiorbit being negative, it must eventually cross $\tilde{\Gamma}_j$ again from right to left; call the first such crossing $\tilde{R}'$. Henceforth, let us drop the primes from the names of these points. Let F_δ (δ positive or negative) be the field constructed form F by rotating through an angle δ. Then the integral curve C_δ of F_δ which passes through $\tilde{Q}$ will pass within a distance ϵ of $\tilde{P}$ and $\tilde{R}$ if $|\delta|$ is small enough. By translation, we may assume that $\tilde{\Gamma}_0$ contans the points $\tilde{P}$, $\tilde{Q}$, and $\tilde{R}$. If $\tilde{Q}$ lies between $\tilde{P}$ and $\tilde{R}$, choose the sign of δ so that C_δ meets $\tilde{\Gamma}_0$ outside the arc $\tilde{P}\tilde{R}$. Replace the arc $\tilde{P}\tilde{R}$ of $\tilde{\Gamma}_0$ by an arc of C_δ, joining so as to form a regular curve $\tilde{\Gamma}'_0$ which is not tangent to F along the newly introduce portion. Likewise define $\tilde{\Gamma}'_i$ for all i. Since the various inverse images of C' do not cross, we may be sure that for δ small enough, the $\tilde{\Gamma}'_i$ do not cross. Hence, their projection Γ' on the torus is a simple closed curve homologous to Γ. By Lemma 4, there are points of tangency on Γ between $\tilde{P}$ and $\tilde{Q}$; since there are none on Γ', the minimal property of Γ is contradicted and the theorem proved for this case. Now suppose $\tilde{R}$ lies between $\tilde{P}$ and $\tilde{Q}$ on $\tilde{\Gamma}_0$. Then there must be another crossing $\tilde{R}_1$. If $\tilde{Q}$, $\tilde{R}$, R_1 are in order on $\tilde{\Gamma}_0$, apply the construction of the first case to them. If not, go on to $\tilde{R}_2$, etc. Now eventually we must come to a situation where the first case can be applied, for if not the semiorbit spirals continually inward, so the limit set would either have to contain or bound a singular point.

(iii) and (iv) are handled similarly to (ii). This proves Lemma 7.

LEMMA 8. *Each closed integral curve C' meets Γ just once and is not tangent at that point.*

Proof. Since the intersection number of C' and Γ is one, they must cross at least once. By Lemma 7, the crossings are not points of tangency. Let $\tilde{P}_0$ on $\tilde{\Gamma}_0$ project into a point of intersection of C' and Γ. Let P_1 on $\tilde{\Gamma}_1$ be a translate of P_0 such that both lie on the boundary of a fundamental domain. Then $\tilde{C}'$ enters it at $\tilde{P}_0$ and leaves at $\tilde{P}_1$. If there is more than one point of intersection, examining the arc $\tilde{P}_0\tilde{P}_1$ reveals that one of the forbidden cases in Lemma 7 occurs.

LEMMA 9. *Let P be a point of tangency on Γ. Then the two semiorbits C' and C'' through P have the same sign and each is asymptotic to a closed integral curve. The arc of Γ containing P and bounded by the points of accumulation of the two semiorbits contains just one point of tangency; its interior meets only open integral curves.*

Proof. Neither semiorbit contains another point of tangency, and neither can cross except toward the side on which it is tangent, by Lemma 7. Hence the two semiorbits have thet same sign. Cutting the torus along C, we get an annulus within which C' and C'' stay; hence, by the Poincaré-Bendixson theorem each approaches a closed integral curve from one side. This curve must be homologous to C. These two curves (both may become C when the annulus is rejoined into a torus) mark off an interval I on Γ. No closed integral curve can meet an interior point of I, since it would separate either C' or C'' from the curve to whcih it is asymptotic. Lift I up into the plane and consider the region bounded by the integral curves through its endpoints. This is divided up into a countable number of subregions by the various inverse images of the integral curve through P. Since each other integral curve lies in one of these regions, all are asymptotic to the boundary curves. None of them can be tangent from the opposite direction to the tangent at P, since otherwise (iv) of Lemma 7 would be contradicted. They cannot be tangent from the same direction, since then Lemma 4 would imply that one of the other curves was tangent from the opposite direction. Hence, there are no other points of tangency on I, and the lemma is proved.

LEMMA 10. *Let C' be an open integral curve not lying in one of the regions constructed in Lemma* 10. *Then the two semiorbits are in opposite directions, and are asymptotic to a pair of closed integral curves which bound a region consisting entirely of open integral curves with semiorbits in opposite directions.*

Proof. Each semiorbit is asymptotic to some closed integral curve, which curves we take to bound an interval I as above. If the semiorbits were in the same direction, some integral curve meeting I would have a point of tangency, which case is excluded. Any other semiorbit is also asymptotic to one of the same two closed curves, by the argument used in Lemma 9.

PROPOSITION 3. *Let Γ have μ^+ points of tangency from the left and μ^- from the right. Then the integral curves may be described as follows:*

(i) *There are μ^+ regions bounded by a pair of (possibly not distinct) closed integral curves, such that all integral curves interior to this region are open and have both semiorbits negative.*

(ii) *There are μ^- such regions with both semiorbits positive.*

(iii) *There are certain other regions composed of open integral curves in which the two semiorbits of a given orbit have opposite signs.*

(iv) *The total number of regions described in* (i), (ii), *and* (iii) *is at most countable.*

(v) *If the complement is not the whole space, it is composed entirely of closed integral curves.*

(vi) *If the complement is the whole space, either all integral curves are closed, or all are dense, or the singular case of Poincaré holds.*

Proof. (i), (ii), (iii) and (v) following immediately from the preceding Lemmas 8, 9, and 10. (iv) follows because these regions are disjoint open sets and the torus has a countable base for open sets. (v) in the case where there exists a cycle without contact, which is dealt with in the references given at the beginning of this section. This proves the proposition.

COROLLARY 4. *Let F be a nonoriented line element field. Then F is orientable if and only if i and j are both even.*

Proof. Since the map of S^1 onto P^1 is of degree 2, the necessity is obvious. For the sufficiency, choose a basis for $H_1(T^2)$ as in the classification theorem. Then by Corollary 2, $\mu = \mu^+ - \mu^-$ is even, and so is $\mu^+ + \mu^-$. Hence there are an even number of regions as in (i) and (ii) and the orientation may be carried out in an obvious manner.

COROLLARY 5. *The number of closed integral curves is at least the greatest common divisor of i and j.*

Proof. The number is at least $\mu^+ + \mu^- \geqq |\mu|$. On the other hand, μ may be described as the greatest common divisor of μ and 0, and the notion of greatest common divisor is invariant under change of basis in $H_1(T^2)$.

5. The Klein bottle. The tangent sphere bundle of the Klein bottle K^2 is not a product, but the tangent projective bundle is. Hence, we may consider nonoriented line element fields as maps $F: K^2 \to P^1$, and define winding numbers as for the torus. The homotopy classes of such maps are determined by the induced homology maps $F_*: H_1(K^2) \to H_1(P^2)$. Since the former group is isomorphic to the direct sum of the integers and the integers modulo 2, these maps are determined by a single integer i which tells how often the free generator covers P^1. Using the covering of the Klein bottle by the torus and the fact that every element of the fundamental group of the Klein bottle is of infinite order, we show that the winding number of a simple closed curve is zero if and only if it is not nullhomotopic. Thus, if F is nonsingular, there can be no closed integral curve which is nullhomotopic.

If an integral curve is homologous to the generator of order 2, cutting along it gives rise to a plane annulus; thus we may show that any other closed integral curve is homologous to it. On the other hand, if $F_* \neq 0$, this is the only homology clas that can contain a closed integral curve. We have proved the following proposition:

PROPOSITION 4. *Let F be a nonoriented line element field on K^2 such that $F_* \neq 0$. Then the only possible closed integral curves are homologous to the generator of order 2 of $H_1(K^2)$.*

Remark. By a theorem of Kneser [3], every line element field on the torus has a closed integral curve. The author would like to see a non-combinatorial proof of the latter theorem.

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NOTE ON FORMAL LIE GROUPS.* [1]

By SHIGEAKI TÔGÔ.

1. Non-commutative formal Lie groups over an algebraically closed field are investigated by J. Dieudonné [3] by considering the homomorphisms of a formal Lie group G into linear groups. When such homomorphisms are injective, the algebraic hull $\mathfrak{A}(G)$ of the image of G by such a homomorphism is defined. Among his results, he asked the questions about the relations between the algebraic hull $\mathfrak{A}(S)$ of a maximal solvable subgroup S (resp. a maximal torus, a Cartan subgroup) of G and a maximal solvable connected subgroup (resp. a maximal torus, a Cartan subgroup) of $\mathfrak{A}(G)$ and between the center of S and that of G, and he also asked whether the l. u. b. of maximal solvable subgroups (resp. Cartan subgroups) of any formal Lie group H is equal to H. It is the purpose of this note to answer these questions and to present some other results on formal Lie groups.

The author wishes to express his thanks to Professor J. Dieudonné for his valuable comments in preparing this note.

2. A formal Lie group G of dimension n over a field K of arbitrary characteristic consists in giving n formal series without constant terms and with coefficients in K in two systems of n indeterminates satisfying the conditions which correspond to the usual condition on the identity and the associative law. Although it is not set-theoretic, all notions corresponding to the set-theoretic notions in groups are introduced into G [3, Chap. I and II]. Through this note we assume that the basic field K is algebraically closed.

We shall recall some of the definitions, results and notations on formal Lie groups given in [3, Chap. III]. We can associate a formal Lie group H^* to any algebraic group H. A formal Lie group G is called *representable* if it is isogenous to a subgroup of the formal Lie group $GL^*(n, K)$ associated to the general linear group $GL(n, K)$. Then the algebraic hull $\mathfrak{A}(G)$ of the image of G by the injective homomorphism of G into $GL^*(n, K)$ can be

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defined. We do not recall the definition. $\mathfrak{A}(G)^*$ is denoted by $\mathfrak{A}^*(G)$. $\mathfrak{A}(G)$ is solvable (resp. nilpotent, commutative) if and only if G is solvable (resp. nilpotent, commutative). If M, N are subgroups of $GL^*(n,K)$ and if M centralizes (resp. normalizes) N, then $\mathfrak{A}(M)$ centralizes (resp. normalizes) $\mathfrak{A}(N)$. It is known that $DG = \mathfrak{A}^*(DG) = D(\mathfrak{A}(G))^*$. If H is a connected algebraic group, then $\mathfrak{A}(H^*) = H$. Let f be a rational homomorphism of H into an algebraic group H_1. Then there exists a corresponding homomorphism $\mathfrak{f}$ of H^* into H_1^* and $\mathfrak{f}(H^*) = f(H)^*$. If N is the kernel of f, then N^* is the kernel of $\mathfrak{f}$. For an element s of H, $\mathfrak{a}_s$ denotes the automorphism of H^* corresponding to the inner automorphism of H induced by s. The subgroups of any formal Lie group form a complete lattice. For its subgroups G_1, G_2, we denote by $G_1 \wedge G_2$, $G_1 \vee G_2$ the g. l. b. and the l. u. b. of G_1 and G_2. If, for algebraic subgroups H_1, H_2 of $GL(n,K)$, we denote by $H_1 \vee H_2$ the smallest algebraic subgroup of $GL(n,K)$ containing H_1, H_2, then we have $(H_1 \vee H_2)^* = H_1^* \vee H_2^*$.

3. We first prove the following

THEOREM 1. *Let G be a representable formal Lie group over an algebraically closed field of arbitrary characteristic. If S_1, S_2 are maximal solvable subgroups (resp. maximal tori, Cartan subgroups) of G, then there exists an element s of $\mathfrak{A}(DG)$ such that $\mathfrak{a}_s(S_1) = S_2$.*

In order to prove this theorem, it suffices to use the following lemma instead of the conjugation theorem by A. Borel in the proofs of [3, Prop. 31 a), Prop. 34 a) and Cor. of Prop. 39].

LEMMA. *Let G be a connected algebraic linear group. If R_1, R_2 are maximal solvable connected subgroups (resp. maximal tori) of G, then there exists an element s of $D^\infty G$ (resp. $C^\infty G$) such that $s^{-1}R_1 s = R_2$.*

Let f be a rational homomorphism of G with $D^\infty G$ as its kernel [2, p. 119]. Then $f(R_1)$ is a maximal solvable connected subgroup of $f(G)$ [1, (22.3)]. Since $f(G)$ is solvable and connected, we have $f(G) = f(R_1)$ and therefore

$$G = R_1(D^\infty G). \tag{1}$$

Let g be the natural mapping of G onto G/R_2. G/R_2 is a projective variety and G operates on G/R_2. Then it is known [1, (15.7)] that R_1 admits a fixed point, which shows that there exists an element s of G such that $R_1 s R_2 = sR_2$, i. e. $s^{-1}R_1 s \subset R_2$. By the maximality of R_1 we have $s^{-1}R_1 s = R_2$.

By virtue of the formula (1) s can be taken in $D^{\infty}G$. The statement for maximal tori follows from the one for maximal solvable connected subgroups and [1, (12.9)].

As an answer to the first question in 1, we shall prove

THEOREM 2. *Let G be a representable formal Lie group over an algebraically closed field of arbitrary characteristic. If S is a maximal solvable subgroup (resp. a maximal torus, a Cartan subgroup) of G, then $\mathfrak{A}(S)$ is a maximal solvable connected subgroup (resp. a maximal torus, a Cartan subgroup) of $\mathfrak{A}(G)$, and conversely.*

It suffices to prove the theorem when G is a subgroup of $GL^*(n,K)$. Let R be a maximal solvable connected subgroup of $\mathfrak{A}(G)$ containing $\mathfrak{A}(S)$. Take a rational homomorphism f of $\mathfrak{A}(G)$ with $D(\mathfrak{A}(G))$ as its kernel [2, p. 119]. Then $f(R)$ is a maximal solvable connected subgroup of $f(\mathfrak{A}(G))$ [1, (22.3)] and therefore $f(R)=f(\mathfrak{A}(G))$. Let $\mathfrak{f}$ be the corresponding homomorphism of $\mathfrak{A}^*(G)$. Then the kernel of $\mathfrak{f}$ is $(D\mathfrak{A}(G))^*=DG$ and we have $\mathfrak{f}(R^*)=f(R)^*$, $\mathfrak{f}(\mathfrak{A}^*(G))=f(\mathfrak{A}(G))^*$. Therefore we have $\mathfrak{f}(R^*)=\mathfrak{f}(\mathfrak{A}^*(G))$ and $\mathfrak{A}^*(G)=R^*\vee DG$. Then we assert that $G=(R^*\wedge G)\vee DG$. In fact, it is clear that $(R^*\wedge G)\vee DG$ is a subgroup of G. Since DG is a normal subgroup of $\mathfrak{A}^*(G)$, by [3, Prop. 11] we have

$$\dim \mathfrak{A}^*(G)=\dim R^*+\dim DG-\dim(R^*\wedge DG).$$

From the facts that $\mathfrak{A}^*(G)=R^*\vee G$ and that G is normal in $\mathfrak{A}^*(G)$, it follows that

$$\dim \mathfrak{A}^*(G)=\dim R^*+\dim G-\dim(R^*\wedge G).$$

Therefore

$$\begin{aligned}\dim G&=\dim(R^*\wedge G)+\dim DG-\dim(R^*\wedge DG)\\&=\dim((R^*\wedge G)\vee DG),\end{aligned}$$

whence we have $G=(R^*\wedge G)\vee DG$, as was asserted. Since $R^*\wedge G$ is a solvable subgroup of G containing S, we have $S=R^*\wedge G$ and

$$G=S\vee DG. \tag{2}$$

Then it is clear that $\mathfrak{A}(G)=\mathfrak{A}(S)\vee\mathfrak{A}(DG)$ and therefore $\mathfrak{A}^*(G)=\mathfrak{A}^*(S)\vee DG$. Since $\mathfrak{A}^*(S)\subset R^*$, we have also $\mathfrak{A}^*(G)=R_0\vee DG$. By using these formulas where DG is normal in $\mathfrak{A}^*(G)$, it is easy to see that

$$\dim R^*=\dim \mathfrak{A}^*(S)+\dim(R^*\wedge DG)-\dim(\mathfrak{A}^*(S)\wedge DG).$$

Since $\mathfrak{A}^*(S) \vee (R^* \wedge DG) \subset R^*$ and $R^* \wedge DG$ is normal in R^*, we have

$$\dim R^* = \dim(\mathfrak{A}^*(S) \vee (R^* \wedge DG))$$

and $R^* = \mathfrak{A}^*(S) \vee (R^* \wedge DG)$. Then we have $R^* = \mathfrak{A}^*(S)$ and therefore $R = \mathfrak{A}(S)$. Conversely, let R_1 be any maximal solvable connected subgroup of $\mathfrak{A}(G)$. If we put $S_1 = G \wedge R_1^*$, then $\mathfrak{A}(S_1) \subset \mathfrak{A}(R_1^*) = R_1$. But S_1 is a maximal solvable subgroup of G by [3, Prop. 31 b)]. Therefore it follows from the direct part that $R_1 = \mathfrak{A}(S_1)$.

Let T be a maximal torus of G and let S be a maximal solvable subgroup of G containing T. Then $\mathfrak{A}(T)$ is a maximal torus of $\mathfrak{A}(S)$ [3, Cor. of Prop. 38]. As is shown above, $\mathfrak{A}(S)$ is a maximal solvable connected subgroup of $\mathfrak{A}(G)$. Therefore $\mathfrak{A}(T)$ is a maximal torus of $\mathfrak{A}(G)$ [1, (16.10)]. The converse part follows from the direct part and [3, Prop. 34 b)].

Let C be a Cartan subgroup of G, i.e. the centralizer $\mathfrak{Z}(T)$ of a maximal torus T in G. Then $\mathfrak{A}(T)$ is a maximal torus of $\mathfrak{A}(G)$. Denote by H the centralizer of $\mathfrak{A}(T)$ in $\mathfrak{A}(G)$. Then H is a Cartan subgroup of $\mathfrak{A}(G)$ [1, (20.4)]. Since $\mathfrak{A}(C)$ centralizes $\mathfrak{A}(T)$, it is clear that $\mathfrak{A}(C) \subset H$. As in the first part of the proof, by considering a rational homomorphism of $\mathfrak{A}(G)$ with $D\mathfrak{A}(G)$ as its kernel, we have $\mathfrak{A}^*(G) = H^* \vee DG$,

$$G = C \vee DG, \tag{3}$$

and then $\mathfrak{A}(C) = H$. Conversely let H_1 be any Cartan subgroup of $\mathfrak{A}(G)$. Then by [1, (20.4)] H_1 is the centralizer of a maximal torus Q of $\mathfrak{A}(G)$. By the statement for a maximal torus, we have $Q = \mathfrak{A}(T_1)$, where T_1 is a maximal torus of G. Put $C = \mathfrak{Z}(T_1)$. Then it follows from the direct part that $H_1 = \mathfrak{A}(C)$. Thus the proof is complete.

From the theorem we have thet following corollaries the first one of which answers the second question in 1.

COROLLARY 1. *Let S be a maximal solvable subgroup of G. Then the center $Z(S)$ of S is equal to the center $Z(G)$ of G.*

By Theorem 2, $\mathfrak{A}(S)$ is a maximal solvable connected subgroup of $\mathfrak{A}(G)$. Therefore $Z(\mathfrak{A}(S)) = Z(\mathfrak{A}(G))$ by [1, (18.5)]. Then we have $Z(\mathfrak{A}^*(S)) = Z(\mathfrak{A}^*(G))$ by [3, Prop. 21]. But we have $Z(S) = G \wedge Z(\mathfrak{A}^*(S))$. Indeed, it is clear that $\mathfrak{A}(S)$ centralizes $\mathfrak{A}(Z(S))$. Then $\mathfrak{A}^*(S)$ centralizes $\mathfrak{A}^*(Z(S))$ and therefore $Z(S)$. It follows that $Z(S) \subset G \wedge Z(\mathfrak{A}^*(S))$. The inverse inclusion is evident. In a similar way it can be verified that $Z(G) = G \wedge Z(\mathfrak{A}^*(G))$. Therefore we have $Z(S) = Z(G)$, completing the proof.

COROLLARY 2. *G is a formal Lie group associated to an algebraic group*

(i) *if and only if a maximal solvable subgroup is associated to an algebraic group*;

(ii) *if and only if a Cartan subgroup is associated to an algebraic group.*

If a maximal solvable subgroup S of G is associated to an algebraic group, then $S = \mathfrak{A}^*(S)$. Using the formula (2) in the proof of Theorem 2, we have

$$G = S \vee DG = \mathfrak{A}^*(S) \vee \mathfrak{A}^*(DG) = (\mathfrak{A}(S) \vee \mathfrak{A}(DG))^*.$$

The converse is evident. The second part can be verified in a similar way by using the formula (3) in the proof of Theorem 2.

The following theorem is an answer to the third question in 1.

THEOREM 3. *Let G be a formal Lie group over an algebraically closed field of characteristic $p > 0$. Then the* l. u. b. *of maximal solvable subgroups (resp. Cartan subgroups) of G is equal to G.*

Put $G' = G/Z(G)$. Then G' is representable [3, Prop. 20]. Let $\mathfrak{f}$ be the natural epimorphism of G onto G'. Then we have $\mathfrak{f}(DG) = DG'$ [3, Cor. of Prop. 16]. Let S be a maximal solvable subgroup of G. Then $\mathfrak{f}(S)$ is a maximal solvable subgroup of G' [3, p. 375]. By the formula (2) in the proof of Theorem 2 we know that $G' = \mathfrak{f}(S) \vee DG'$. Therefore we have $\mathfrak{f}(S \vee DG) = G'$. Since $Z(G) \subset S$, we have $G = S \vee DG$. Denote by G_1 the l. u. b. of maximal solvable subgroups of G. As is shown in [3, p. 376], owing to a result of A. Borel, the l. u. b. of maximal solvable subgroups of G' contains $D(G')$. Therefore G_1 contains DG and we have $G_1 = G$.

Let T be a maximal torus of G and put $T' = \mathfrak{f}(T)$. Then it is known that T' is a maximal torus of G' [3, p. 379]. Put $C = \mathfrak{Z}(T)$. Then $\mathfrak{f}(C)$ is contained in $\mathfrak{Z}(T')$. It is clear that $\mathfrak{f}^{-1}(\mathfrak{Z}(T'))$ is nilpotent and contains C. By the maximality of C we have $C = \mathfrak{f}^{-1}(\mathfrak{Z}(T'))$ and therefore $\mathfrak{f}(C) = \mathfrak{Z}(T')$, i. e. $\mathfrak{f}(C)$ is a Cartan subgroup of G'. By the formula (3) in the proof of Theorem 2 we know that $G' = \mathfrak{f}(C) \vee DG'$. Therefore we have $\mathfrak{f}(C \vee DG) = G'$. Since C contains $Z(G)$, we have $G = C \vee DG$. It is now easy to conclude that the l. u. b. of Cartan subgroups of G is equal to G.

4. By [3, Cor. 2 of Prop. 9 and Prop. 11c)] and the chain conditions we see that the l. u. b. N of all solvable normal subgroups of a formal Lie group G is the largest solvable normal subgroup of G. We call N the *radical* of G.

G is called *semi-simple* if the radical of G is equal to e (cf. [3, p. 385]). We shall prove the following

THEOREM 4. *Let G be a representable formal Lie group over an algebraically closed field of arbitrary characteristic and let N be the radical of G. Then $\mathfrak{A}(N)$ is the radical of $\mathfrak{A}(G)$, and $G \wedge \mathfrak{A}^*(N) = N$.*

It is clear that $\mathfrak{A}(N)$ is a solvable normal subgroup of $\mathfrak{A}(G)$. Therefore, if we denote by M the radical of $\mathfrak{A}(G)$, then $\mathfrak{A}(N)$ is contained in M. Let f be a rational homomorphism of $\mathfrak{A}(G)$ with M as its kernel. Then $f(\mathfrak{A}(G))$ is a connected semi-simple algebraic group. By using the results of C. Chevalley that any connected semi-simple algebraic group is equal to its derived subgroup, we have

$$f(D^\infty \mathfrak{A}(G)) = D^\infty f(\mathfrak{A}(G)) = f(\mathfrak{A}(G)).$$

The rest of the proof of the first part of the theorem runs exactly parallel to the corresponding part in the proof of Theorem 2. It can therefore be omitted. We only note that, coresponding to the formulas (2) and (3), we have

$$G = N \vee D^\infty G. \tag{4}$$

the second part is evident.

COROLLARY 1. *A representable formal Lie group is associated to an algebraic group if and only if so is the radical.*

This can be proved in the same manner as in the proof of Corollary 2 of Theorem 2, by making use of the formula (4).

Furthermore, the fact stated in [3, Th. 7] is another consequence of the theorem. Namely we have

COROLLARY 2. *If G is a semi-simple formal Lie group, then G is isogenous to a group H^* where H is a semi-simple algebraic group, and conversely.*

If G is a semi-simple Lie group, then $Z(G) = e$ and therefore G is representable. There exists an isogenous homomorphism $\boldsymbol{u}$ of G such that $\boldsymbol{u}(G)$ is a subgroup of $GL^*(n, K)$. It now follows from Corollary 1 that $\boldsymbol{u}(G) = H^*$ where H is an algebraic group. By the above theorem it is obvious that the radical of H is equal to e, i.e. H is semi-simple. Conversely, suppose that H' is a connected semi-simple algebraic group. If N is the

radical of H'^*, then $\mathcal{A}(N)$ is the radical of H' since $\mathcal{A}(H'^*) = H'$. Therefor $\mathcal{A}(N) = e$, whence $N = e$, i.e. H'^* is semi-simple, completing the proof.

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ON SEMISIMPLICIAL FIBRE-BUNDLES.*

By M. G. BARRATT,[1] V. K. A. M. GUGENHEIM [2] and J. C. MOORE.[1]

I. 1. Introduction. The intent of this paper is to begin the study of semisimplicial fibre bundles, and to show how general fibre bundles are. The semisimplicial analogue of a fibre space of the type defined by Serre has been known for several years now, and it has been realized that if one replaced a geometric fibre map of this type by the associated map of the singular complex of the total space into the singular complex of the base one obtains a semisimplicial fibre map, or Kan fibre map. However, one can go much further. Here, given a Kan fibre map, we first choose a "minimal fibre map" (cf. II. 2) which is a fibre-wise deformation retract of the original fibre map, and then proceed to show that this minimal fibre map is in fact the map of the total complex of a semisimplicial fibre bundle into the base complex.

In the description of a semisimplicial fibre-bundle with fibre Y there occurs a certain group-complex $A(Y)$ or, more generally, one of its subgroups Γ; the consequential notion "Γ-bundle" is the semisimplicial analogue of a Steenrod fibre bundle with structural group Γ (see the definitions in IV. 2. 4); in particular the classical notions of the associated principal bundle and the classifying space reappear in our theory (IV. 4. 3, 4. 4, 5. 6).

Applications of this theory to the relationship between fibre-bundles and fibre-space (in the classical sense), and to the homology theory of fibre-spaces, will be treated elsewhere.

I. 2. Terminology. The terminology and theory of semisimplicial complexes (cf. [1], for instance) will be taken for granted; "complex" and "map" will mean "complete semisimplicial complex" and "complete semisimplicial map" respectively. The symbol 1 will denote the identity (in many contexts), the unit element in certain groups, as well as an object described in II. 2 below.

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II. Apparatus. This chapter is purely technical and deals mostly with the treatment of homotopies in complexes; cf. [2] for more details.

1. Models. The n-simplex Δ^n is, for our purposes, best defined as follows: Let δ^n be a "generator" in dimension n; Δ^n is the set of all elements $\phi\delta^n$, where ϕ is a semisimplicial operator with domain-dimension n.

We shall write

$$\delta^n{}_i = \partial_0 \cdots \partial_{i-1}\partial_{i+1} \cdots \partial_n \delta^n \epsilon \Delta^n{}_0$$

and call it the "i-th vertex."

Let X be any complex. We shall frequently identify the element $x \in X_n$ with the map $x\colon \Delta^n \to X$ defined by $x(\phi\delta^n) = \phi x$; in particular we define

$$\epsilon^i = \partial_i\delta^n\colon \Delta^{n-1} \to \Delta^n, \qquad \eta^i = s_i\delta^n\colon \Delta^{n+1} \to \Delta^n.$$

In Δ^n we define a "-contraction operator" D by

$$D\phi\delta^n = \phi' s_0\delta^n,$$

where ϕ' is obtained from ϕ by increasing all subscripts by 1. We have

$$\begin{aligned} &\partial_0 D = \text{identity},\\ &\partial_{i+1}D = D\partial_i \text{ in dimension } > 0,\\ &\partial_1 Dx = \delta^n{}_0 \text{ if } x \in \Delta^n{}_0,\\ &D\delta^n{}_0 = s_0\delta^n{}_0. \end{aligned} \tag{1}$$

Definition. We call a complex B simplicially contractible (to b_0) if there is an operator D satisfying (1) with $\delta^n{}_0$ replaced by b_0.

2. Prisms. We write $I = \Delta^1$; also, it will be convenient to denote δ'_0, δ'_1 and all their degeneracies by 0, 1 respectively.

Let A be a complex; by $A_{r,n}$ we denote the set of all maps $u\colon I^r \times \Delta^n \to A$ (where $I^1 = I$, $I^{r+1} = I \times I^r$). In an evident manner we can define operators

$$\begin{aligned} &\partial_i\colon A_{r,n} \to A_{r,n-1} && (0 \leqq i \leqq n),\\ &s_i\colon A_{r,n} \to A_{r,n+1} && (0 \leqq i \leqq n),\\ &\lambda^\epsilon{}_j\colon A_{r,n} \to A_{r-1,n} && (\epsilon = 0, 1, 1 \leqq j \leqq r),\\ &\sigma_j\colon A_{r,n} \to A_{r+1,n} && (1 \leqq j \leqq r+1), \end{aligned}$$

where we identify $A_{0,n}$ and A_n in the obvious way; the operators $\lambda^\epsilon{}_j$ are obtained by taking the face $\partial_{1-\epsilon}$ in the j-th factor; the ∂_i, s_i satisfy the usual relations; the $\lambda^\epsilon{}_j$, σ_j satisfy the corresponding identities of the well-known "semicubical theory"; finally, "simplicial" and "cubical" operators commute.

There are also "simplicial subdivisionoperators"

$$S_i\colon A_{r,n}\to A_{r-1,n+1} \qquad (0\leqq i\leqq n)$$

satisfying certain identities (cf. [2], [3]); an element $u\in A_{r,n}$ is completely determined by the elements S_iu.

3. Homotopies.

3.1. LEMMA. *Let $f_0, f_1\colon A\to B$ be maps. The following two statements are equivalent:*

(i) *There is a map $F\colon I\times A\to B$ such that*

$$F(0,x)=f_0(x), \qquad F(1,x)=f_1(x).$$

(ii) *There is an operator $U\colon A_q\to B_{1,q}$ $(q\geqq 0)$ such that*

$$\lambda^0{}_1U=f_0, \qquad \lambda^1{}_1U=f_1, \qquad \partial_iU=U\partial_i, \qquad s_iU=Us_i.$$

In fact, we define

$$U(x)(\phi\delta^1,\psi\delta^q)=F(\phi\delta^1,\psi x),$$
$$F(\phi\delta^1,x)=U(x)(\psi\delta^1,\delta^q).$$

where ϕ, ψ are semisimplicial operators.

Definition. In the situation of 3.1 we call f_0, f_1 homotopic, F or U a homotopy.

If B is a Kan complex, homotopy is an equivalence relation.

Note. In questions of homotopy, we shall usually use the "classical" form (i) in the statements of our propositions; nearly always, however, it is the form (ii) that is best used in the proofs.

We say that a space B is contractible (to b_0) if the maps $1\colon B\to B$ and $B\to b_0\in B_0$ are homotopic; this means that there is a homotopy $H\colon B\to B$ such that

$$\lambda^1{}_1H=1,$$
$$\lambda^0{}_1Hx=s^q{}_0b_0 \quad \text{if} \quad x\in B_q,$$
$$\partial_iH=H\partial_i,$$
$$s_iH=Hs_i,$$
$$Hb_0=\sigma_1b_0.$$

It is easily proved (using the operators S_i!) that if a space is simplicially contractible, then it is contractible.

III. Fibre Maps.

1. Terminology. A map $p: E \to B$ will be denoted by (E, B, p) and a commutative diagram

$$\begin{array}{ccc} E' & \xrightarrow{\tilde{f}} & E \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{f} & B \end{array}$$

by $(\tilde{f}, f): (E', B', p') \to (E, B, p)$; we refer to $(\tilde{f}, f)$ as a map.

Two maps $(\tilde{f}_0, f_0)$, $(\tilde{f}_1, f_1): (E', B', p') \to (E, B, p)$ will be said to be *homotopic* if there is a map $(\tilde{F}, F): (I \times E', I \times B', 1 \times p') \to (E, B, p)$ such that $\tilde{F} \mid 0 \times E' = \tilde{f}_0$, $\tilde{F} \mid 1 \times E' = \tilde{f}_1$ (with a slight abuse of language!), and similarly for F. If $f_0 = f_1 = f$ and $F = fp_2$, where $p_2: I \times B \to B$ is the projection, then we say that $(\tilde{f}_0, f)$ and $(\tilde{f}_1, f)$ are homotopic *rel. f*.

Homotopy is alternatively expressed by the existence of $\tilde{U}: E'_n \to E_{1,n}$, $U: B'_n \to B_{1,n}$ $(n \geqq 0)$ such that $p\tilde{U} = Up'$ and $\lambda^0{}_1\tilde{U} = \tilde{f}$, $\lambda^1{}_1 U = \tilde{f}_1$, $\lambda^0{}_1 U = f_0$, $\lambda^1{}_1 U = f_1$. The homotopy is rel. f if $U = \sigma_1 f$.

If there are maps $(\tilde{f}, f)$ and $(\tilde{g}, g)$ such that $(\tilde{f}\tilde{g}, fg)$, $(\tilde{g}\tilde{f}, gf)$ are homotopic to $(1, 1)$, then we call either a *homotopy equivalence*; if $f = g =$ identity on B, and the two homotopies are rel. this identity, then we refer to a *strong* homotopy equivalence; in particular we have the notion of *homotopy retraction* and *strong homotopy retraction*.

2. Definition. (E, B, p) will be called a *fibre-map* if p is onto and for $n > 0$ satisfies the following condition:

Given an integer t, $0 \leqq t \leqq n$, $e_i \in E_{n-1}$ for all i, $0 \leqq i \leqq n$, $i \neq t$, and $b \in B_n$ such that

$$\partial_t e_j = \partial_{j-1} e_i \qquad (i < j, i, j \neq t),$$

$$pe_i = \partial_i b \qquad (i \neq t),$$

then there is $e \in E_n$ such that

$$e_i = \partial_i e \qquad (i \neq t),$$

$$b = pe.$$

If, further, $\partial_t e$ depends on the e_i and b only, then (E, B, p) is said to be a *minimal* fibre-map.

3. Homotopy covering and lifting theorems.

3.1. LEMMA. *Let* $L = 0 \times \Delta^{n-1} \cup I \times \dot{\Delta}^{n-1} \subset I \times \Delta^{n-1}$, *let* (E, B, p) *be a fibre-map and*

$$\bar{F}: I \times \Delta_{n-1} \to B,$$

$$F_0: L \to E$$

be such that $pF_0 = \bar{F} \mid L$.

Then F_0 *has an extension* $F: I \times \Delta^{n-1} \to E$ *such that* $pF = \bar{F}$.

If (E, B, p) *is minimal,* $F \mid 1 \times \Delta^{n-1}$ *is unique.*

In other words (cf. 2.2), *given* $e_i \in E_{1,n-1}$ $(n > 0, 0 \leqq i \leqq n)$, $e^0{}_1 \in E_n$, $b \in B_{1,n}$ *such that*

$$\partial_i e_j = \partial_{j-1} e_i \qquad (i < j),$$

$$\lambda^0{}_1 e_i = \partial_i e^0{}_1,$$

$$pe_i = \partial_i b, \qquad pe^0{}_1 = \lambda^0{}_1 b,$$

then there is $e \in E_{1,n}$ *such that*

$$e_i = \partial_i e, \qquad e^0{}_1 = \lambda^0{}_1 e, \qquad pe = b.$$

If p *is minimal,* $\lambda^1{}_1 e$ *is unique. The roles of* $\lambda^0{}_1$, $\lambda^1{}_1$ *can be exchanged.*

This is a special case of Theorem 3* in [2], except for the addendum which follows easily from the argument there given.

3.2. Covering homotopy extension theorem. *In the commutative diagram* $(\bar{f}, f): (E', B', p) \to (E, B, p)$, *det* p *be a fibre-map, let* $A' \subset B'$ *be a subcomplex* (*possibly empty!*) *and* $G' = p'^{-1}A' \subset E'$.

Let maps

$$H: I \times B' \to B,$$

$$\tilde{h}: I \times G' \to E$$

be given such that

$$H(0, b') = f(b'), \qquad b' \in B',$$

$$\tilde{h}(0, g') = \bar{f}(g'), \qquad g' \in G',$$

$$p\tilde{h} = Hp' \mid G'.$$

Then h *can be extended to* $\tilde{H}: I \times E' \to E$ *such that*

$$\tilde{H}(0, e') = fe', \qquad p\tilde{H} = H(1 \times p').$$

This can be "translated" into the prism language and then proved, using 3.1 and a familiar argument.

By a classical argument (cf. [4], p. 54), we deduce

3.3. Lifting homoptopy extension theorem. *Let (E,B,p) be a fibre-map, $A' \subset B'$ complexes and*

$$f: B' \to E, \qquad H: I \times B' \to B, \qquad F_A: I \times A' \to E$$

maps such that

$$pf(b') = H(0,b'), \qquad pF_A = H \mid I \times A',$$
$$F_A(0,a') = f(a') \qquad (a' \in A', b' \in B').$$

Then F_A has an extension $F: I \times B' \to E$ such that $F(0,b') = f(b')$ and $pF = H$.

4. Retraction on a minimal fibre-map. (cf. [1]).

4.1. THEOREM. *Let (E,B,p) be a fibre-map; there exists a subcomplex $E' \subset E$ such that (E',B,p'), where $p' = p \mid E'$, is a minimal fibre-map and a strong deformation retract of (E,B,p).*

4.2. THEOREM. *Let (E,B,p) be a fibre-map and $B^* \subset B$ a given minimal deformation retract of B; there exists $E^* \subset E$ such that (E^*,B^*,p^*), where $p^* = p \mid E^*$, is minimal and a deformation retract of (E,B,p), the deformation retraction lying over the given one.*

Outline of the proof of 4.1. We call $e, e' \in E_n$ *p-compatible* if $pe = pe'$, $\partial_i e = \partial_i e'$ for $0 \leqq i \leqq n$. p-compatible elements e, e' are said to be *p-homotopic* if there is an element $u \in E_{1,n}$ such that

$$pu = \sigma_1 pe = \sigma_1 pe',$$
$$\lambda^0{}_1 u = e, \qquad \lambda^1{}_1 u = e',$$
$$\partial_i u = \sigma_1 \partial_i e = \sigma_1 \partial_i e'$$

If p is a fibre-map, p-homotopy is an equivalence-relation; degenerate p-homotopic elements are equal; and, most important, p is minimal if and only if p-compatible and p-homotopic elements are equal.

The proof of 4.1 consists in building up E' dimension by dimension always choosing one element only in each p-homotopy class; elements having boundary not already in E' we first "move" into ones that have, using the inductively defined retraction on the boundaries.

The proof of Theorem 4.2 uses a succession of homotopy-covering arguments.

It will appear from 5.2 below that the minimal map obtained in Theorem 4.1 is unique up to an "equivalence."

5. Homotopy and minimal fibre spaces. In view of the last section we can, for the purposes of homotopy-theory, replace any fibre-map by a minimal one; this procedure is useful largely because of the theorems of the present section.

A map $f: A \to B$ will be called an *injection* if $fa = fa'$ implies $a = a'$; an injection onto will be called an *equivalence.*

5.1. PROPOSITION. *Let* $(\bar{f}_0, f_0), (\bar{f}_1, f_1): (E', B', p') \to (E, B, p)$ *be homotopic maps, and let* p *be a minimal fibre map.*

(i) *If* $\bar{f}_0$, f_1 *are injective, so is* $\bar{f}_1$.

(ii) *Let* $E^\epsilon = p^{-1}(f_\epsilon B')$ $(\epsilon = 0, 1)$. *If* $\bar{f}_0$, f_0, f_1 *are injections onto* E^0, f_0B', f_1B' *respectively, then* $\bar{f}_1$ *is an injection onto* E^1.

Proof. We are given homotopies $\bar{H}: E' \to E$, $H: B' \to B$ such that $p\bar{H} = Hp'$ and $\lambda^0{}_1\bar{H} = \bar{f}_0$, $\lambda^1{}_1H = \bar{f}_1$, $\lambda^0{}_1H = f_0$, $\lambda^1{}_1H = f_1$.

(i) Let $e', e'' \in E'_n$ be such that $\bar{f}_1e' = \bar{f}_1e''$. Then $p\bar{f}_1e' = p\bar{f}_1e''$, i.e. $f_1p'e' = f_1p'e''$, whence $p'e' = p'e'' = b'$ say. Let $w' = \bar{H}e'$, $w'' = \bar{H}e''$. Then $pw' = pw'' = Hb'$, $\lambda^1{}_1w' = f_1e' = f_1e'' = \lambda^1{}_1w''$. If $n = 0$, minimality implies $\lambda^0{}_1w' = \lambda^0{}_1w''$, i.e., $\bar{f}_0e' = \bar{f}_0e''$ and $e' = e''$. If $n > 0$, we use an induction on n and thus have $\partial_ie' = \partial_ie''$ for $0 \leqq i \leqq n$; hence $\partial_iw' = \partial_iw''$, and the same conclusion can be made.

(ii) $\bar{f}_1$ is injective, by (i). Let $e \in E^1{}_n$, $b = pe$. Then there is $b' \in B'$ such that $f_1b' = b$, i.e. $b = \lambda^1{}_1Hb'$.

Hence, if $n = 0$, there is $w \in E_{1,0}$ such that $pw = Hb'$, $\lambda^1{}_1w = e$. Then $\lambda^0{}_1w \in E^0$ and there is $e' \in E'$ such that $f_0e' = \lambda^0{}_1w$. Then

$$f_0p'e' = p\bar{f}_0e' = p\lambda^0{}_1w = \lambda^0{}_1pw = \lambda^0{}_1Hb' = f_0b',$$

whence $p'e' = b'$. Now:

$$p\bar{H}e' = Hp'e' = Hb' = pw,$$

$$\lambda^0{}_1He' = \bar{f}_0e' = \lambda^0{}_1w.$$

Hence $\lambda^1{}_1He' = f_0e' = \lambda^0{}_1w$, by the condition of minimality, i.e. $\bar{f}_1e' = e$, and $\bar{f}_1$ is onto $E^1{}_0$.

For $n > 0$, we use an induction on n; we write down suitable conditions for $\partial_i w$; and in this we require that $\bar{f}_1$ should be injective. We omit the details.

5.2. COROLLARY. *A strong homotopy equivalence between minimal fibre-maps is a strong equivalence. A homotopy equivalence between minimal fibre-maps with minimal base-spaces is an equivalence.*

5.3. *Definition.* Let (E, B, p) and (A, B, f) be maps; we define the map (E^*, A, p^*) and the map $(\bar{f}, f) : (E^*, A, p^*) \to (E, B, p)$ as follows:

$$E^* = \{(e, a) \in E \times A \mid fa = pe\},$$

$$p^*(e, a) = a, \qquad \bar{f}(e, a) = e.$$

(E^*, A, p^*) is called the *map induced from* (E, B, p) *by* $f : A \to B$; the entire assembly of maps and spaces in $(\bar{f}, f)$ is called the *induced structure*

5.4. LEMMA. *If (E, B, p) is a (minimal) fibre-map, so is any map induced from it.*

5.5. PROPOSITION. *Let (E, B, p) be a minimal fibre-map and $f_0, f_1 : A \to B$ homotopic maps. Then the induced maps (E^0, A, p^0), (E^1, A, p^1) are strongly equivalent.*

Proof. Let $F : I \times A \to B$ be the given homotopy and $(E^*, I \times A, p^*)$ the minimal fibre space induced by F. $p^{*-1}(\epsilon \times A)$ $(\epsilon = 0, 1)$ can be indentified with E^ϵ. Now, consider the commutative diagram

$$\begin{array}{ccc} I \times E^0 & \xrightarrow{\phi} & E^* \\ {\scriptstyle 1 \times p^0}\downarrow & & \downarrow{\scriptstyle p^*} \\ I \times A & \xrightarrow[1]{} & I \times A, \end{array}$$

where $\phi \mid 0 \times E^0$ is the identity; such a map ϕ exists by 3.2. Now,

$$\phi^0 = \phi \mid 0 \times E^0 \qquad (= 1),$$

$$\phi^1 = \phi \mid 1 \times E^0$$

can be regarded in a natural manner as maps $E^0 \to E^*$; they satisfy the conditions on f_0, f_1 in 5.1 (ii). Hence ϕ^1 is an isomorphism $E^0 \to E^1$.

5.6. COROLLARY. *A minimal fibre space with a contractible base-space is a product space.*

IV. Fibre Bundles.

1. The group complex $A(Y)$. Let X, Y be complexes. By X^Y (cf. [1], [2]) we denote the complex whose n-elements are maps

$$\alpha\colon \Delta^n \times Y \to X,$$

and we define $\partial_i\alpha = \alpha(\epsilon^i \times 1)$, $s_i\alpha = \alpha(\eta^i \times 1)$ (cf. II.1). Sometimes it is convenient to replace α by the map $\alpha_I\colon \Delta^n \times Y \to \Delta^n \times X$ given by $\alpha_I(t,y) = (t, \alpha(t,y))$, $t \in \Delta^n$, $y \in Y$. Clearly α and α_I determine each other.

Let X, Y, Z be complexes, and let $\alpha \in X^Y$, $\beta \in Y^Z$; we define $\alpha\beta \in X^Z$ by $(\alpha\beta)_I = \alpha_I\beta_I$. It is easily verified that

$$\partial_i(\alpha\beta) = (\partial_i\alpha)(\partial_i\beta), \qquad s_i(\alpha\beta) = (s_i\alpha)(s_i\beta).$$

The complex Y^Y is a monoid under this operation; it operates on Y according to

$$\alpha \cdot y = \alpha(\delta^n, y), \qquad \alpha \in (Y^Y)_n, \qquad y \in Y_n.$$

We define $A(Y) \subset Y^Y$ to be the maximal subgroup; thus $\alpha \in Y^Y$ is in $A(Y)$ if and only if α_I has an inverse. $A(Y)$ is a group-complex and hence (cf. [1]) a Kan-complex.

$A(Y)$ has an important universal property: Let Γ be any group-complex operating on Y; we define the homomorphism

$$\rho\colon \Gamma \to A(Y)$$

by $(\rho\gamma)(\phi\delta^n, y) = (\phi\gamma)y$, where $\gamma \in \Gamma$, $y \in Y$ and ϕ is a suitable semisimplicial operator. Then $\gamma y = (\rho\gamma) \cdot y$. If Γ is effective, i.e., if ρ is a monomorphism, we shall thus regard Γ as a subgroup of $A(Y)$, identifying y and ρy; replacing (where necessary) Γ by $\Gamma/\text{kernel}\,\rho$, *we restrict ourselves to this case from now on.*

2. Fibre bundles.

2.1. *Definition.* The map (E, B, p) will be called a *fibre-bundle* if

(i) p is onto.

(ii) For every map $b\colon \Delta^n \to B$ the induced map (E^b, Δ^n, p^b) is strongly equivalent to $(\Delta^n \times Y, \Delta^n, p^*)$, where $p^*(\phi\delta^n, y) = \phi\delta^n$ and Y is a given complex called the *fibre* of the bundle.

If Y is a Kan-complex, we call (E, B, p) a *Kan fibre-bundle.*

From III. 5.6 we have immediately

2.2. PROPOSITION. *Every minimal fibre map with a connected base complex is a Kan fibre bundle.*

The following is easy:

2.3. PROPOSITION. *Every Kan fibre bundle is a fibre map.*

Let (E, B, p) be a fibre-bundle with fibre Y; regarding $b \in B_n$ as a map $b: \Delta^n \to B$ (cf. II.1) we have the commutative diagram

$$\begin{array}{ccccc} & & \alpha(b) & & \bar{b} \\ \Delta^n \times Y & & \longrightarrow & E^b & \longrightarrow & E \\ \downarrow & p^* & & \downarrow & p^b & \downarrow & p \\ \Delta^n & & \longrightarrow & \Delta^n & \longrightarrow & B, \\ & & 1 & & b & \end{array}$$

where $\alpha(b)$ is an equivalence exhibiting the strong equivalence of 2.1(ii). The set of equivalences $\{\alpha(b)\}$ for $b \in B$ is called an *atlas* for the bundle. We also define

$$\beta(b) = \bar{b}\alpha(b).$$

Observe that $\alpha(b)(\phi\delta^n, y) = (\phi\delta^n, \beta(b)(\phi\delta^n, y))$ so that the atlas $\{\alpha(b)\}$ and the set of maps $\{\beta(b)\}$ determine each other.

The choice of an atlas is highly arbitrary; if $\{\alpha(b)\}$, $\{\bar{\alpha}(b)\}$ are two atlases, then

$$(\alpha(b))^{-1}\bar{\alpha}(b) = \gamma_I(b)$$

is an element of $A(Y)_n$. Conversely, if for every $b \in B_n$ we choose $\gamma(b) \in A(Y)_{n-1}$ and if $\{\alpha(b)\}$ is a given atlas, then $\{\bar{\alpha}(b)\}$ given by

$$\bar{\alpha}(b) = \alpha(b)\gamma_I(b) \tag{1}$$

is another. Notice that also

$$\bar{\beta}(b) = \beta(b)\gamma(b). \tag{2}$$

Given an atlas $\{\alpha(b)\}$, $\beta(b) \in E^Y$ for every $b \in B$. In general it is not true that

$$\beta(s_i b) = s_i \beta(b). \tag{3}$$

By the simple device, however, of defining $\beta(b)$, and hence $\alpha(b)$, on all degenerate elements by (3) above we can always replace an atlas by a *normalised atlas*, i.e., one for which (3) above is true. From now on we shall invariably suppose atlases to be normalised.

More interesting considerations arise from the face operators. Let $b \in B_n$. $\alpha(b)$ maps $\Delta^{n-1} \times Y$ isomorphically onto the part of E^b "over" $\delta_i\delta^n$; hence there is an isomorphism $\alpha_i(b) : \Delta^{n-1} \times Y \to E^{\delta_i b}$ such that the diagram

$$\begin{array}{ccccc} \Delta^n \times Y & \xrightarrow{\alpha(b)} & E^b & \xrightarrow{\tilde{b}} & E \\ \uparrow{\scriptstyle \epsilon^i \times 1} & & \uparrow{\scriptstyle \tilde{\epsilon}^i \times 1} & & \uparrow{\scriptstyle 1} \\ \Delta^{n-1} \times Y & \xrightarrow[\alpha_i(b)]{} & E^{\partial_i b} & \xrightarrow[\widetilde{\partial_i b}]{} & E \end{array}$$

commutes ($0 \leqq i \leqq n$). The important thing is that, in general, $\alpha_i(b) \neq \alpha(\partial_i b)$, i.e., $\partial_i \beta(b) \neq \beta(\partial_i b)$. We define

$$\xi^i{}_I(b) = [\alpha(\partial_i b)]^{-1} \alpha_i(b) \in A(Y)_{n-1} \tag{4}$$

and refer to $\{\xi^i(b)\}$ as the set of *transformation elements* associated with the atlas $\{\alpha(b)\}$.

From (4) we get immediately

$$\alpha_i(b) = \alpha(\partial_i b)\xi^i{}_I(b), \tag{5}$$

whence

$$\partial_i \beta(b) = \beta(\partial_i b)\xi^i(b). \tag{6}$$

2.4. *Definitions.* Let $\Gamma \subset A(Y)$ be a subgroup-complex. An atlas $\{\alpha(b)\}$ all of whose transformation-elements lie in Γ will be called a Γ-*atlas*; two Γ-atlases $\{\alpha(b)\}$, $\{\bar{\alpha}(b)\}$ will be called Γ-*equivalent* if $\bar{\alpha}(b) = \alpha(b)\gamma_I(b)$, where $\gamma(b) \in \Gamma$. A fibre-bundle together with a given Γ-equivalence-class of Γ-atlases will be called a Γ-*bundle*; thus any bundle with fibre Y is an $A(Y)$-bundle.

Let (E, B, p), $(\bar{E}, \bar{B}, p)$ be Γ-bundles; a map $(f, g) : (E, B) \to (\bar{E}, \bar{B})$ will be called a Γ-*map* if for every $b \in B$

$$[\bar{\alpha}(fb)]^{-1} \circ f \circ \alpha(b)$$

is an element of Γ, provided that $\{\alpha(b)\}$ and $\{\bar{\alpha}(\bar{b})\}$ belong to the given Γ-equivalence classes of atlasses; notice the consequential notions of "Γ-equivalence" and "strong Γ-equivalence"; for bundles with fibre Y, "equivalence" and "$A(Y)$-equivalence" are the same notions.

From now on we will formulate everything for the general case of Γ-bundles.

Let $\Lambda \subset \Gamma$ be a subgroup-complex; if in the given Γ-equivalence-class of atlases of a given Γ-bundle there is a Γ-bundle there is a Λ-atlas, we say that

the *group of the bundle can be reduced to* Λ; notice, however, that the resulting Λ-bundle will, in general, not be unique.

2.5. LEMMA. *In every* Γ*-equivalence class of atlases there is one for which* $\xi^i(b) = 1$ *for* $i > 0$.

Proof. Let $b \in B_n$. If $n = 1$, since Γ (being a group-complex) is a Kan-complex we can find $\gamma \in \Gamma_1$ such that $\partial_1\gamma = \xi^1(b)$. We replace $\alpha(b)$ by $\bar{\alpha}(b) = \alpha(b)\gamma^{-1}{}_I$. Then $\partial_1\beta(b) = \beta(\partial_1 b)\xi^1(b)[\xi^1(b)]^{-1} = \beta(\partial_1 b)$.

Now suppose inductively that $\{\alpha(b)\}$ satisfies the lemma up to dimension $n-1$, $n \geqq 2$, and let $b \in B_n$ be non-degenerate. From the inductive hypothesis it is easy to verify $\partial_i\xi^j(b) = \partial_{j-1}\xi^i(b)$ if $0 < i < j$. Hence there is $\gamma \in \Gamma_n$ such that $\partial_i\gamma = \xi^i(b)$ for $i < 0$. We replace $\alpha(b)$ by $\bar{\alpha}(b) = \alpha(b)\gamma^{-1}{}_I$. Then, for $i > 0$, $\partial_i\bar{\beta}(b) = \beta(\partial_i b)\xi^i(b)[\xi^i(b)]^{-1} = \beta(\partial_i b)$.

2.6. LEMMA. *Let* $\Lambda \subset \Gamma$ *be a subcomplex which is a deformation retract of* Γ. *Then in every* Γ*-equivalence class of atlases there is one for which*

$$\xi^i(b) = 1, \qquad i > 0,$$
$$\xi^0(b) \in \Lambda.$$

Proof. Let $N : \Gamma \to \Gamma$ be the given retracting homotopy, so that $\lambda^0{}_1 U = 1$, $\lambda^1{}_1 U\gamma \in \Lambda$, $U \mid \Lambda = \sigma_1$. Let the given atlas $\{\alpha(b)\}$ already satisfy the conditions of 2.5 and, inductively, $\xi^0(b) \in \Lambda$ if $\dim b < n$. Now let $b \in B_n$ and $\xi = \xi^0(b)$. Then it is easily verified that $\partial_i\xi \in \Lambda$ for all i. By the Kan-condition (and cf. [2]) there is a prism $T \in \Gamma_{1,n}$ such that

$$\lambda^0{}_1 T = s_0\xi, \qquad \partial_i T = \sigma_1\partial_i s_0\xi \qquad (i > 0),$$
$$\partial_0 T = U\xi,$$

since these faces are consistent. Now, let $\gamma = (s_0\xi)^{-1}(\lambda^1{}_1 T)$. Then, as is easily verified,

$$\partial_i\gamma = 1 \qquad (i > 0),$$
$$\partial_0\gamma = \xi^{-1}\lambda,$$

where $\lambda = \lambda^1{}_1 U\xi \in \Lambda$.

Now, replace $\alpha(b)$ by $\bar{\alpha}(b) = \alpha(b)\gamma_I$, so that $\bar{\beta}(b) = \beta(b)\gamma$. Then $\partial_i\bar{\beta}(b) = \beta(\partial_i b)$, $i > 0$, and $\partial_0\bar{\beta}(b) = \beta(\partial_0 b)\xi\xi^{-1}\lambda = \beta(\partial_0 b)\lambda$.

3. Twisted Cartesian products.

3.1. *Definition.* Let B, Y be complexes. A *twisted cartesian product*

(TCP) $B \widetilde{\times} Y$ is a complex for which $(B \widetilde{\times} Y)_n = B_n \times Y_n$ $(n \geqq 0)$ and

$$\partial_i(b, y) = (\partial_i b, \partial_i y) \qquad (0 < i \leqq n, n > 0),$$
$$\partial_0(b, y) = (\partial_0 b, \tau(b, y)), \qquad b \in B_n, y \in Y_n,$$

where $\tau(b, y) \in Y_{n-1}$.

τ is called the *twisting function.*

From the semisimplicial identities the following identities follow for τ:

$$\begin{aligned} \tau(\partial_1 b, \partial_1 y) &= \tau(\partial_0 b, \tau(b, y)), \\ \partial_i \tau(b, y) &= \tau(\partial_{i+1} b, \partial_{i+1} y), && i > 0, \\ s_i \tau(b, y) &= \tau(s_{i+1} b, s_{i+1} y), \\ \tau(s_0 b, s_0 y) &= y. \end{aligned} \tag{1}$$

The twisted cartesian product (and τ) are called *semi-regular* if $\tau(b, y)$ depends on b and $\partial_0 y$ only.

We refer to a *regular* TCP (RTCP) if for every $b \in B_n$ $(n > 0)$, there is an element $\xi(b) \in A(Y)_{n-1}$ such that

$$\tau(b, y) = \xi(b) \cdot \partial_0 y.$$

If for every $b \in B$, $\xi(b)$ lies in a subgroupcomplex Γ of $A(Y)$ the RCTP is said to *have group* Γ.

By virtue of (1), the function ξ must satisfy the following identities

$$\begin{aligned} \partial_0 \xi(b) &= [\xi(\partial_0 b)]^{-1} \xi(\partial_1 b), \\ \partial_i \xi(b) &= \xi(\partial_{i+1} b), && i > 0, \\ s_i \xi(b) &= \xi(s_{i+1} b), && i \geqq 0, \\ \xi(s_0 b) &= 1. \end{aligned} \tag{2}$$

Let $E = B \widetilde{\times} Y$; we define the map $p: E \to B$ by $p(b, y) = b$; from now on, with a slight abuse of language, the term "TCP" will also stand for the map $(B \widetilde{\times} Y, B, p)$.

3.2. PROPOSITION. *A fibre bundle whose group can be reduced to* Γ *is (strongly equivalent to) a* RTCP *with group* Γ.

Proof. Let (E, B, p) be a fibre bundle with fibre y and group Γ, and let $\{\alpha(b)\}$ be a Γ-atlas satisfying the conditions of 2.5 above, we define

$$\xi(b) = \xi^0(b), \qquad b \in B_n, n \geqq 1$$

and the functions

$$h: B_n \times Y_n \to E_n$$

by

$$h(b, y) = \beta(b)(\delta^n, y).$$

Then the following are easily verified:

(i) h is $1-1$ and onto; it is easy to write down an inverse;

(ii) $$\partial_i h(b, y) = h(\partial_i b, \partial_i y) \qquad (i > 0), \qquad \text{and}$$

$$\partial_0 h(b, y) = h(\partial_0 b, \xi(b) \cdot \partial_0 y)\text{; as well as}$$

$$s_i h(b, y) = h(s_i b, s_i y).$$

Notice that once the atlas is given, h is determined without further choice; thus, if we define a Γ-map between two RTCP $B \widetilde{\times} Y$, $B' \widetilde{\times} Y$, both with group Γ, as one having the form $(b, y) \to (gb, \theta(b) \cdot y)$, where $g: B \to B'$ is a map and $\theta: B \to \Gamma$ a certain function; then the corresponding notion of a Γ-equivalence class of RTCP corresponds, under the function h, exactly to that of a Γ-equivalence class of Γ-bundles, cf. 2.4 above. We omit the easy verifications.

3.3. Proposition. *An* RTCP *whose fibre is a Kan-complex is a Kan fibre-space.*

The verification is easy.

4. Principal and associated bundles. Let Γ be a group complex; in what follows we shall always suppose that Γ operates on itself from the left by multiplication; this leads to a natural embedding $\Gamma \subset A(\Gamma)$.

4.1. *Definition.* A Γ-bundle with fibre Γ is called a *principal* Γ-*bundle.*

If (E, B, p) is a principal Γ-bundle, it is possible to define an operation of Γ on the *right* of E satisfying $p(e\gamma) = p(e)$, as follows:

Let $\{\alpha(b)\}$ be a Γ-atlas in the Γ-equivalence-class of atlases defining the given Γ-bundle, let $e \in E$, $pe = b$, and

$$e = \beta(b)(\delta^n, \gamma).$$

Then we define

$$e\gamma' = \beta(b), \delta^n, \gamma\gamma').$$

It is easily verified that this *does* define an operation which is independent of the particular Γ-atlas chosen.

If we express a principal Γ-bundle as a RCTP, then $\xi(b) \in \Gamma$, and the operation of Γ takes the simple form

$$(b, \gamma)\gamma' = (b, \gamma\gamma').$$

Let Γ operate on the right of the complex E and on the left of the complex Y; we define $E \times_\Gamma Y$ by identifying $(e\gamma, y)$ and $(e, \gamma y)$ in $E \times Y$.

4.2. *Definition.* Let (E, B, p) be a given principal Γ-bundle and Y a complex on which Γ operates; we define the *associated bundle* (E^*, B, p^*) by

$$E^* = E \times_\Gamma Y, \qquad p^*(e, y) = pe.$$

It is easily verified that (E^*, B, p^*) is a bundle with fibre Y; indeed, let $\{\alpha(b)\}$ be an atlas for (E, B, p); then we define the atlas $\{\alpha^*(b)\}$ for (E^*, B, p^*) by

$$\beta^*(b)(\phi\delta^n, y) = (\beta(b)(\phi\delta^n, 1), y).$$

In this way we see that

4.3. PROPOSITION. *For a given principal Γ-bundle and a given complex Y on which Γ operates on the left, there is a unique associated Γ-bundle with fibre Y.*

4.4. PROPOSITION. *Every Γ-bundle with fibre Y is associated to a principal Γ-bundle which is uniquely determined.*

Proof. We take the Γ-bundle in the form of a RCTP with twisting function $\xi(b)$; the principal RTCP with the same twisting function is the required principal Γ-bundle; the fact that any RTCP *is* a fibre bundle is proved in 5.3 below.

5. Classification theorems. Let us use the notation of III. 5.5 for the structure induced from the map (E, B, p) by the map $f: A \to B$.

According as (E, B, p) *is* a fibre-bundle, Γ-bundle or principal bundle, so is (E^*, A, p^*); in particular the atlases are related by

$$\tilde{f}\beta^*(a) = \beta(fa);$$

it is in this way that a Γ-equivalence-class of atlases for (E, B, p) determines one for (E^*, A, p^*); in the case of principal bundles we have

$$\tilde{f}(e^*\gamma) = [\tilde{f}(e^*)]\gamma,$$

i.e. $\tilde{f}$ is equivariant.

If E is a RTCP, $E = B \widetilde{\times} Y$, then so is E^*; the twisting function is given by

$$\xi^*(a) = \xi(fa).$$

Now, let (E, B, p) be a principal Γ-bundle and $U: A \to B$ a homotopy; we write $\lambda^i{}_1 U = f_i$; $(i = 0, 1)$, and denote by

$$(\bar{f}_i, f_i) : (E_i, A, p_i) \to (E, B, p)$$

the induced structures.

For the next lemma notice that if a multiplication is defined amongst elements of complexes, then this implies one between prisms.

5.1. LEMMA. *There exists a homotopy*

$$V: E_0 \to E$$

such that

$$pV = Up_0,$$

$$\lambda^0{}_1 V = \bar{f}_0,$$

$$V(e_0\gamma) = (Ve_0)(\sigma_1\gamma) \qquad (e_0 \in E_0, \gamma \in \Gamma).$$

Proof. This is proved exactly like III. 3.2, except having "lifted" V to some chosen $e_0 \in p_0^{-1}(b)$, we define it on the rest of $p_0^{-1}(b)$ by equivalence, i.e., by $V(e_0\gamma) = (Ve_0)(\sigma_1\gamma)$; this is possible since $p_0^{-1}(b) = e_0\Gamma$, as is evident.

Now, let us suppose that (E, B, p) is expressed as a RSTP $B \widetilde{\times} \Gamma$ with twisting function ξ. Then

$$\lambda^1{}_1 V(a, \gamma) = (f_1 a, \theta(a)\gamma),$$

where $\theta(a)$ is defined by $(\lambda^1{}_1 V)(a, 1) = (f_1 a, \theta(a))$. The function $\theta: A \to \Gamma$ satisfies certain identities.

5.2. PROPOSITION. *Homotopic maps induce Γ-equivalent bundles from a given Γ-bundle.*

Proof. Take the given Γ-bundle in the form of a RTCP $B \times Y$ with twisting function ξ. Let $U: A \to B$ be the given homotopy, and construct $\theta: A \to \Gamma$ as above. Then

$$(a, y) \to (a, \theta(a) \cdot y)$$

gives the equivalence. Several verifications are left to the reader.

Notice that the proof only uses the fact that we have a RTCP; hence we see that any RTCP which has a contractible base-space is a product space; in particular

5.3. PROPOSITION. *Any* RTCP *is a fibre-bundle.*

Remark. A semi-regular TCP is a fibre-bundle provided it is a Kan fibre-map; we omit the somewhat complicated proof.

For any group-complex Γ, the well-known "W-construction" provides a classifying space $\bar{W}(\Gamma)$; we recall briefly the definition, which we modify by inverting the usual order of elements; this is done to assure operation on the right.

Given Γ, we define the RTCP $(W(\Gamma), \bar{W}(\Gamma), p)$ as follows:

$$W_0(\Gamma) = \Gamma_0,$$
$$W_{q+1}(\Gamma) = W_q(\Gamma) \times \Gamma_{q+1},$$

and Γ_q acts on $W_q(\Gamma)$ by the definition

$$(\gamma_0, \gamma_1, \cdots, \gamma_q) \cdot \gamma'_q = (\gamma_0, \gamma_1, \cdots, \gamma_{q-1}, \gamma_q \cdot \gamma'_q),$$
$$\text{where } \gamma_i \in \Gamma_i, \gamma'_q \in \Gamma_q.$$

The semisimplicial operators are defined as follows: s_0 on $W_0(\Gamma) = s_0$ on Γ_0. Now, let $w \in W_q(\Gamma)$ and $\gamma \in \Gamma_{q+1}$, so that $(w, \gamma) \in W_{q+1}(\Gamma)$; then we define inductively

$$\partial_0(w, \gamma) = w \cdot \partial_0\gamma,$$
$$\partial_{i+1}(w, \gamma) = (\partial_i w, \partial_{i+1}\gamma),$$
$$s_0(w, \gamma) = (w, 1, s_0\gamma),$$
$$s_{i+1}(w, \gamma) = (s_i w, s_{i+1}\gamma).$$

Next, we define $\bar{W}(\Gamma)$ as follows:

$\bar{W}_0(\Gamma)$ consists of a single element denoted by [ ],

$$\bar{W}_{q+1}(\Gamma) = \bar{W}_q \times \Gamma_q.$$

Thus, if $w \in W_q(\Gamma)$, we denote by $[w]$ the corresponding element of $\bar{W}_{q+1}(\Gamma)$.

We define $p: W(\Gamma) \to \bar{W}(\Gamma)$ by $p(w, \gamma) = [w]$, $p\gamma = [\ ]$ if $\gamma \in \Gamma_0$. Suitable semisimplicial operators in $\bar{W}(\Gamma)$ are now easily written down; it turns out that $W(\Gamma) = \bar{W}(\Gamma) \tilde{\times} \Gamma$ is a principal RTCP with twisting function

$$\xi[\gamma_0, \gamma_1, \cdots, \gamma_{q-1}] = \gamma_{q-1}.$$

Now, let $E = B \widetilde{\times} \Gamma$ be a principal RTCP with twisting function ξ. We define the map

$$k_\xi : B \to \bar{W}(\Gamma)$$

by

$$k_\xi(b) = [\xi(\partial^{n-1}{}_0 b), \xi(\partial^{n-2}{}_0 b), \cdots, \xi(b)] \quad \text{if} \quad b \in B_n.$$

5.4. LEMMA. (i) *k_ξ is a map.*

(ii) *k_ξ induces the given* RTCP *from* $(W(\Gamma), \bar{W}(\Gamma), p)$.

The verifications are straightforward.

5.5. PROPOSITION. *The assignment $\xi \to k_\xi$ set up a* 1-1 *relationship between homotopy-classes of maps $B \to \bar{W}(\Gamma)$ and strong Γ-equivalence-classes of principal Γ-bundles with base-complex B.*

Proof. Homotopic maps induce equivalent bundles, by 5.3. It remains to prove the converse. Thus, let $k_0, k_1 : B \to \bar{W}(\Gamma)$ be maps with induced structures $(\tilde{k}_\epsilon, k_\epsilon) : (E_\epsilon, B, p_\epsilon) \to (W(\Gamma), \bar{W}(\Gamma), p)$, $\epsilon = 0, 1$; and suppose there is a strong Γ-equivalence $(\tilde{k}, 1) : (E_0, B, p_0) \to (E_1, B, p_1)$.

Consider the maps $\tilde{k}_0, \tilde{k}_1\tilde{k} : E \to W(\Gamma)$. We shall construct an equivariant homotopy connecting them; i.e., a homotopy

$$V : E_0 \to W(\Gamma)$$

such that $\lambda^0{}_1 V = \tilde{k}_0$, $\lambda^1{}_1 V = \tilde{k}_1\tilde{k}$,

$$V(e_0\gamma) = (V(e_0))\sigma_1\gamma.$$

Suppose, inductively, V has been defined up to dimension $n-1$ and let $e_0 \in (E_0)_n$ be some chosen element. An element $V(e_0)$ satisfying the conditions exists because $W(\Gamma)$ has trivial homotopy-groups; we define $V(e_0\gamma)$ by the condition of equivariance.

Now, we define a homotopy $U : B \to \bar{W}(\Gamma)$ by choosing for each $b \in B_n$ an $e_0 \in (E_0)_n$ such that $p_0 e_0 = b$; and take $U(b) = pV(e_0)$; due to the condition of equivariance on V, U is well-defined. Now,

$$\lambda^0{}_1 U(b) = p\tilde{k}_0(e_0) = k_0 p_0 e_0 = k_0 b,$$

$$\lambda^1{}_1 U(b) = p\tilde{k}_1\tilde{k}(e_0) = k_1 p_1 k e_0 = k_1 p_0 b_0 = k_1 b,$$

and thus U is the required homotopy.

5.6. THEOREM. *Let Y be a complex on which the group complex Γ operates effectively. The assignment to any map $B \to \bar{W}(\Gamma)$ of the bundle*

with fibre Y associated to the principal Γ-bundle over B induced from $(W(\Gamma), \overline{W}(\Gamma), p)$ is a 1-1 *correspondence between homotopy-classes of maps $B \to \overline{W}(\Gamma)$ and strong Γ-equivalence-classes of Γ-bundles with base B and fibre Y.*

Notice, in particular, the special case obtained by putting $\Gamma = A(Y)$, and replacing "Γ-bundle" by "bundle" "Γ-equivalence" by "equivalence."

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SINGULAR INTEGRALS ON COMPACT MANIFOLDS.*

By R. T. Seeley.[1]

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Introduction. Singular integrals have been studied intermittently for over twenty years, on both Euclidean space and on manifolds, primarily by Giraud, Tricomi, Mihlin, Calderon and Zygmund, and Kohn. The work of the first three authors is summarized in [6]. Calderon and Zygmund have

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considered these operators on the spaces $L^p(E_k)$, $(1 < p < \infty)$, E_k being k-dimensional Euclidean space. In this case they have shown that singular integrals define bounded operators [1], that there is an approximate functional calculus for such operators resembling the Fourier transform apparatus [2], and that they may be used to represent differential operators [2]. This representation has been used in solving uniqueness problems [3] and is likely to have further applications in differential equations.

The present paper obtains for compact manifolds results like those in [2]. Kohn, in a paper with Spencer [5], has considered similar problems but is concerned mainly with the question of regularization, and the fitting of singular integral operators into the formalism of differential geometry. Here the two main objectives are the formulation of the notion of a singular integral operator on a compact manifold, and the development of an approximate functional calculus (section II); and the construction of an isotropic first order differential operator on a compact, orientable Riemannian manifold M allowing the representation of differential operators on $L^p(M)$ (section III). Section I establishes the notation and contains the Euclidean results used in the later sections, as well as a few related results. In the construction in section III, singular integral operators are used to establish some facts about the Laplace operator on $L^2(M)$ previously given by Gaffney [4].

The results labeled "proposition" are those that are easily accessible with the tools developed, but do not form part of the main argument.

This paper is a revision of the author's thesis of the same title submitted to MIT in 1958. Naturally it owes much to the supervisor, Professor Calderon, who suggested the general topic, the consideration of the distribution spaces, the present formulation of section II without a metric (considerably neater than the original form), and that the operators needed in section III might be constructed by contour integration; and whose comments on earlier versions led to much improvement in the presentation. The author, on the other hand, is responsible for the actual proofs.

I. Miscellaneous Euclidean Results

A. Notation and preliminary definitions. Let E_k be k-dimensional Euclidean space and $\alpha = (\alpha_1, \cdots, \alpha_k)$ a k-tuple of non-negative integers. If $x = (x_1, \cdots, x_k)$ is a point in E_k, then $x^\alpha = \prod_{i=1}^{k} x_i^{\alpha_i}$. Let $|\alpha| = \sum_i \alpha_i$ and $\alpha! = \prod_i (\alpha_i!)$. Then $(\partial/\partial x)^\alpha = \prod_i (\partial/\partial x_i)^{\alpha_i} = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_k^{\alpha_k}$. $f^{(\alpha)}(x)$

denotes $(\partial/\partial x)^\alpha f(x)$; so $(1/\alpha!)(x^\alpha)^{(\beta)}$ is $x^{\alpha-\beta}/(\alpha-\beta)!$ if $\alpha_i \geqq \beta_i$ for all i $(\alpha \geqq \beta)$, and is 0 otherwise. The Taylor expansion becomes

$$f(y) = \sum_{0 \leqq |\alpha| \leqq n} (1/\alpha!) f^{(\alpha)}(x)(y-x)^\alpha + R(x,y).$$

Here the remainder can be differentiated with respect to y as often as f can. The rule for differentiating products takes the form

$$(fg)^{(\alpha)} = \sum_{\alpha \geqq \gamma} f^{(\gamma)} g^{(\alpha-\gamma)} \alpha!/\{\gamma!(\alpha-\gamma)!\}.$$

For a function $f(x,y)$ on $E_k \times E_k$, $f^{(\alpha,\beta)}(x,y)$ denotes $(\partial/\partial x)^\alpha (\partial/\partial y)^\beta f(x,y)$. Differentiation of the Taylor expansion above shows that $R^{(\alpha,\beta)}(x,y)$ is $O(|x-y|^{n+1-|\alpha|-|\beta|})$ if $|\alpha|+|\beta| \leqq n+1$.

If h is a complex function, h^* is its conjugate; if H operates on a Banach space, H^* is its adjoint.

If β is a real number $\geqq 0$, then $[\beta]$ indicates either the greatest integer which is $\leqq \beta$, or a reference to the bibliography. A function f is in C_n if it and its derivatives of order $\leqq n$ are continuous and bounded. A function is in C_β if it and its derivatives of order $\leqq [\beta]$ satisfy a uniform Hölder condition of order $\beta - [\beta]$.

Definition 1. A C_β^∞ function homogeneous of degree n is a function $h(x,z)$ on $E_k \times E_k$ satisfying:

i) $h(x,\lambda z) = \lambda^n h(x,z)$ for all x, all $z \neq 0$, and all $\lambda > 0$.

ii) For each x, $h(x,z)$ is in C_∞ in $|z| \geqq 1$.

iii) For each α, $(\partial/\partial z)^\alpha h(x,z)$ is in C_β on $E_k \times (|z| \geqq 1)$.

iv) If $n = -k$, $\int_{|z|=1} h(x,z)\,d\sigma = 0$, where $d\sigma$ is the natural measure on $|z| = 1$.

Condition (iv) may seem unnatural, but in this paper we shall always require it of homogeneous functions of degree $-k$. It is justified to some extent by the following fact.

LEMMA 1. *If $h(x,z)$ is a C_β^∞ function homogeneous of degree n, then $(\partial/\partial z_i)h(x,z)$ is a C_β^∞ function homogeneous of degree $n-1$.*

Proof. The degree of homogeneity is trivial. The only case that needs checking is $n = 1-k$; then we must show that $h_i(x,z) = (\partial/\partial z_i)h(x,z)$ has mean value zero on $|z| = 1$. Let $\phi(y) = 1$ on $|x-y| \leqq 1$, and vanish outside a compact set. Then

$$\int (\partial\phi(y)/\partial y_i)h(x,x-y)\,dy = \lim_{\epsilon\to 0}\int_{|x-y|=\epsilon}(\partial\phi(y)/\partial y_i)h(x,x-y)\,dy$$

$$= \lim_{\epsilon\to 0}\int_{|x-y|>\epsilon}\phi(y)h_i(x,x-y)\,dy - \lim_{\epsilon\to 0}\int_{|x-y|=\epsilon}\phi(y)h(x,x-y)\gamma_i\epsilon^{k-1}\,d\sigma,$$

where $d\sigma$ is the measure on the unit sphere, $y=x+\epsilon\sigma$, and $\gamma_i=(y_i-x_i)/|y-x|$ is the i-th direction cosine. The last limit is $-\int_\Sigma h(x,\sigma)\gamma_i\,d\sigma$, where Σ is the unit sphere. Then $\lim_{\epsilon\to 0}\int_{|x-y|>\epsilon}\phi(y)h_i(x,x-y)\,dy$ must exist. But this is

$$\lim_{\epsilon\to 0}\int_{1\geqq|x-y|>\epsilon}h_i(x,x-y)\,dy + \int_{|x-y|>1}\phi(y)h_i(x,x-y)\,dy$$

$$= \lim_{\epsilon\to 0}\int_\Sigma h_i(x,\sigma)\,d\sigma\int_\epsilon^1(1/\rho)\,d\rho + \int_{|x-y|>1}\phi(y)h_i(x,x-y)\,dy.$$

This can have a limit only if the first term is zero, which proves Lemma 1.

Homogeneous C_β^∞ functions of degree $-k$ are the kernels of the C_β^∞ singular integral operators studied by Calderon and Zygmund ([1] and [2]). A C_β^∞ singular integral operator is any of the form

$$Hf(x) = a(x)f(x) + \lim_{\epsilon\to 0}\int_{|x-y|>\epsilon}h(x,x-y)f(y)\,dy,$$

where $h(x,z)$ is a C_β^∞ function, homogeneous of degree $-k$, and $a(x)$ is in C_β. The limit exists strongly for f in L^p, $1<p<\infty$ [1]. In [2] the above authors develop an approximate functional calculus for such operators, and demonstrate a connection to differential equations.

An important tool in this development is the expansion of h in spherical harmonics. Let $Y_{nm}(z)$ $(|z|=1)$ be a spherical harmonic of degree n, and let the set $\{1, Y_{nm}(z)\}$ form a complete orthonormal basis of L^2 of the unit sphere in E_k. Extend Y_{nm} to $|z|>0$ by setting $Y_{nm}(z)=Y_{nm}(z/|z|)$. Then if $h(x,z)$ is a C_β^∞ function homogeneous of degree $-k$,

$$h(x,z) = \sum a_{nm}(x)Y_{nm}(z)\,|z|^{-k}$$

and the a_{nm} and their derivatives of order $\leqq[\beta]$ are $O(n^{-r})$ for every $r>0$. The operators T_{nm} are defined by

$$T_{nm}f(x) = \lim_{\epsilon\to 0}\int_{|x-y|>\epsilon}Y_{nm}(x-y)f(y)\,dy,$$

and the operator H is represented by the series $H=a+\sum a_{nm}T_{nm}$. The operators T_{nm} correspond to multiplication of the Fourier transform by a bounded function, so that they commute with differentiation. It is noted that

the number of Y_{nm} for fixed n is $O(n^{k-2})$, and the norm of T_{nm} on L^p has a bound independent of n and m so that the series is norm convergent. (See [2].)

B. The fractional integrals of M. Riesz in connection with singular integrals. In [7], M. Riesz constructs, among other things, a one-parameter group of (unbounded) operators I^α defined for f in C_∞ with compact support in E_k. They satisfy the relations $I^{-2} = -\Delta$, where Δ is the Laplace operator on E_k; and $I^{\alpha+\beta} = I^\alpha I^\beta$ whenever the composition is defined. For $\alpha > 0$, I^α is defined as

$$I^\alpha f(x) = H(\alpha, k)^{-1} \int f(y) \mid x - y \mid^{\alpha-k} dy,$$

where $H(\alpha, k) = 2^\alpha \pi^{k/2} \Gamma(\alpha/2)/\Gamma(\frac{1}{2}(k-\alpha))$. For $\alpha \leqq 0$, $I^\alpha f(x)$ is defined by analytic continuation with respect to α. Since I^{-1} is a square root of $-\Delta$, it seems reasonable to show its relation to the operator Λ defined in [2]; this operator is defined by $\Lambda = i \sum (\partial/\partial x_m) R_m$, where R_m is the Riesz transform given by

$$R_m f(x) = -i\Gamma(s)\pi^{-s} \lim_{\epsilon \to 0} \int_{\mid x-y \mid > \epsilon} (x_m - y_m) \mid x - y \mid^{-k-1} f(y)\, dy, \quad s = \tfrac{1}{2}(k+1).$$

The Fourier transform of R_m is $x_m/\mid x \mid$, so that the transform of Λ is $\mid x \mid$. We show that $(\partial/\partial x_m) I^1 = iR_m$, from which it follows that $I^{-1} = -\Lambda$:

$$\Lambda = i \sum (\partial/\partial x_m) R_m = \sum (\partial/\partial x_m)^2 I^1 = -I^{-2} I = -I^{-1}.$$

To prove $(\partial/\partial x_m) I^1 = iR_m$, let g and f be in C_1 and have compact support. Then

$$-(\partial g/\partial x_m, I^1 f) = -H(1,k)^{-1} \int \partial g(x)/\partial x_m \int f^*(y) \mid x - y \mid^{1-k} dy\, dx$$

$$= -H^{-1} \int f^*(y) \lim_{\epsilon \to 0} \int_{\mid x-y \mid > \epsilon} (\partial g(x)/\partial x_m) \mid x - y \mid^{1-k} dx\, dy$$

$$= (1-k) H^{-1} \int f^*(y) \lim_{\epsilon \to 0} \int_{\mid x-y \mid > \epsilon} g(x)\,(x_m - y_m) \mid x - y \mid^{-k-1} dx\, dy$$

$$+ H^{-1} \int f^*(y) \lim_{\epsilon \to 0} \int_{\mid x-y \mid = \epsilon} g(x) \gamma_m\, d\sigma_x\, dy;$$

here γ_m is the m-th direction cosine and $d\sigma_x$ the measure on the unit sphere, and $x = y + \epsilon\sigma$. Since γ_m has mean value 0 and g is in C_1, the second limit in the last expression is zero, and $-(\partial g/\partial x_m, I^1 f) = (Rg, f)$, where

$$Rg = (1-k) H(1,k)^{-1} \lim_{\epsilon \to 0} \int_{\mid x-y \mid > \epsilon} g(x)\,(x_m - y_m) \mid x - y \mid^{-k-1} dx = -iR_m g.$$

Hence $-(\partial g/\partial x_m, I^1 f) = -i(R_m g, f) = (g, iR_m f)$, which is what we were to show.

C. The Green's kernel of $\Delta - \lambda$. We calculate the Green's kernel $E_\lambda(|x-y|)$ of $\Delta - \lambda$ by setting $r = |x-y|$, $E_\lambda(|x-y|) = y(r)$. Then y satisfies

$$y''(r) + \frac{k-1}{r} y'(r) - \lambda y(r) = 0, \qquad r > 0.$$

Let

$$k = 2\nu + 2, \ \lambda = \mu^2 \qquad (\mathrm{Re}(\mu) \geqq 0), \ y(r) = r^{-\nu} Y(r).$$

The equation for Y is $r^2 Y'' + rY' - (\nu^2 + r^2\mu^2) Y = 0$, which has the singular solution $Y(r) = AK_\nu(\mu r)$, K a modified Bessel function of the second kind. If ν is an integer, $\nu = n$,

$$\begin{aligned} K_\nu(z) = \tfrac{1}{2} \sum_{m=0}^{n-1} (-1)^m (n-m-1)\,!/\{m\,!(z/2)^{n-2m}\} \\ + (-1)^{n+1} \sum_{m=0}^{\infty} \{\log(z/2) - (\tfrac{1}{2})\psi(m+1) \\ - (\tfrac{1}{2})\psi(n+m+1)\}\,(z/2)^{n+2m}/\{m\,!(n+m)\,!\}. \end{aligned}$$

If ν is $n + \frac{1}{2}$, n an integer, then

$$K_\nu(z) = (\pi/2z)^{\frac{1}{2}} e^{-z} \sum_{r=0}^{n} (n+r)\,!/\{r\,!(n-r)\,!(2z)^r\}$$

([8], p. 80). We choose the constant A in order to have the Green's function for $\Delta - \lambda$, and define

$$E_\lambda(r) = Ar^{-\nu} K_\nu(\mu r), \text{ where } \nu = (\tfrac{1}{2})k - 1, \tag{1}$$

$A = -(\mu/2)^\nu / \nu\,! \omega_k$ or $A = -1/\omega_2$ according as $k > 2$ or $k = 2$, and ω_k is the area of the unit sphere in E_k. That this is indeed a Green's kernel for $\Delta - \lambda$ can be checked by applying Green's formula to

$$\int_{|x-y|>\epsilon} E_\lambda(|x-y|)\,(\Delta - \lambda) f(y)\, dy.$$

From the series for K_ν, we get the following estimates for E_λ:

$$\begin{aligned} &|\, dE_\lambda/dr - \omega_k^{-1} r^{1-k} \,| \leqq C\,|\,\mu\,|\, r^{2-k} \text{ or } C\,|\,\mu \log \mu r\,| \text{ in } |\,\mu r\,| \leqq 1 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{according as } k > 2 \text{ or } k = 2, \\ &|\, d^2E_\lambda/dr^2 - r^{-k}(1-k)\omega_k^{-1} \,| \leqq C\,|\,\mu\,|\, r^{1-k} \text{ in } |\,\mu r\,| \leqq 1, \\ &d^n E_\lambda/dr^n = A' r^{2-n-k} + O(r^{3-n-k}) \qquad\qquad (r \to 0) \end{aligned} \tag{2}$$

generally for fixed λ.

For global estimates consider the integral representation of $K_\nu(z)$ ([8], p. 172):

$$K_\nu(z) = \Gamma(\nu+\tfrac{1}{2})^{-1}\Gamma(\tfrac{1}{2})\,(\tfrac{1}{2}z)^\nu \int_1^\infty e^{-zt}(t^2-1)^{\nu-\frac{1}{2}}\,dt$$

so that, replacing t by $1 + t/\mu r$,

$$E_\lambda(r) = \mu^{2\nu}C\int_1^\infty e^{-\mu r t}(t^2-1)^{\nu-\frac{1}{2}}\,dt = C'e^{-\mu r}r^{2-k}\int_0^\infty e^{-t}[t(t+2\mu r)]^{\nu-\frac{1}{2}}\,dt,$$

where C and C' depend only on k. If we set

$$I_\lambda(r) = \int_0^\infty e^{-t}t^{\frac{1}{2}(k-3)}(t+2\mu r)^{\frac{1}{2}(k-3)}\,dt$$

with $\mu = \lambda^{\frac{1}{2}}$, $\mathrm{Re}(\mu) \geqq 0$, we have

$$d^nI_\lambda/dr^n = \int_0^\infty e^{-t}2^{-n}(k-3)\cdots(k-2n-1)t^{\frac{1}{2}(k-3)}(2\mu)^n\ (t+2\mu r)^{\frac{1}{2}(k-2n-3)}\,dt.$$

If $n \leqq \frac{1}{2}(k-3)$, this is in absolute value $\leqq |\mu|^n P(|\mu|r)$, where P is a polynomial (depending on n and k) with positive coefficients. If k is odd, this holds for all n.

If k is even, $n \geqq \frac{1}{2}(k-3)$, and $r\mu \geqq 0$, then $|d^nI_\lambda/dr^n|$ does not exceed

$$2^{-n}\int_0^\infty e^{-t}\,|(k-3)\cdots(k-2n-1)|\cdot t^{\frac{1}{2}(k-3)}(2|\mu|)^n(2|\mu|r)^{\frac{1}{2}(k-3)-n}\,dt$$
$$\leqq Ar^{-n}(|\mu|r)^{\frac{1}{2}(k-3)}.$$

These estimates lead immediately to

$$(3)\qquad \begin{aligned} &|d^nE_\lambda/dr^n| \leqq |e^{-\mu r}|\,r^{2-k-n}P_n(|\mu|r) \text{ if } n \leqq \tfrac{1}{2}(k-3) \text{ or } k \text{ is odd};\\ &|d^nE_\lambda/dr^n| \leqq e^{-\mu r}r^{2-k-n}(\mu r)^{\frac{1}{2}(k-3)}P_n(\mu r) \text{ if } n \geqq \tfrac{1}{2}(k-3) \text{ and } \mu \geqq 0, \end{aligned}$$

where P_n is a polynomial with constant coefficients depending on k and n.

We can use these kernels to construct operators equivalent to the I^α of the previous section for $\alpha \geqq 0$; this will motivate a similar construction on a Riemannian manifold to be given later. Let $\delta > 0$, $k > 2$, and

$$J_\delta^\alpha = (\delta-\Delta)^{-\alpha/2} = \tfrac{1}{2\pi i}\int_C \lambda^{-\alpha/2}(\lambda-\delta+\Delta)^{-1}\,d\lambda = \tfrac{1}{2\pi i}\int_C \lambda^{-\alpha/2}E_{\delta-\lambda}\,d\lambda,$$

where C is the path $\mathrm{Re}(\lambda) = \delta/2$ traversed from top to bottom, and the cut for $\lambda^{-\alpha/2}$ is taken along $\lambda \leqq 0$. From the estimates (3), E_λ is a bounded operator on $L^p(E_k)$, and

$$\|E_\lambda\| \leqq \omega_k\int_0^\infty |E_\lambda(r)|\,r^{k-1}\,dr \leqq \omega_k\sum a_m\int_0^\infty |e^{-\mu r}|\,|\mu r|^m\,r\,dr < A\,|\lambda|^{-1},$$

($\mathrm{Re}(\lambda) \geqq 0$, $k > 2$) so the integral defining J_δ^α converges in norm if $\alpha > 0$. Since $E_\lambda = (\Delta - \lambda)^{-1}$, its Fourier transform is $-(\lambda + |x|^2)^{-1}$, and the transform of J_δ^α is

$$\hat{J}_\delta^\alpha(x) = \tfrac{1}{2\pi i}\int_C \lambda^{-\alpha/2}(-|x|^2 - \delta + \lambda)^{-1}d\lambda = (|x|^2 + \delta)^{-\alpha/2},$$

where the root is positive. Thus the J_δ^α form a semi-group of bounded operators on L^2, with $J_\delta^2 = (\delta - \Delta)^{-1}$. J_δ^α converges strongly to the identity as $\alpha \to 0$, since $\hat{J}_\delta^\alpha$ converges boundedly to 1.

We can eliminate the dependence of C on δ, for $0 < \alpha < 2$, by transforming the contour into one surrounding the negative real axis. With this contour Γ we can use the estimates (3) for $k = 2$, and find again $J_\delta^\alpha = (\delta - \Delta)^{-\alpha/2}$, $0 < \alpha < 2$.

From the expression for $\hat{J}_\delta^1$, we see that if f has compact support and is continuous, $J_\delta^1 f$ converges uniformly to $-I^1 f$, where $-I^1$ is the inverse of Λ, discussed in the previous section. Thus $\frac{1}{2\pi i}\int_\Gamma \lambda^{-\frac{1}{2}}E_{-\lambda}(r)\,d\lambda = -H(1,k)^{-1}r^{1-k}$, or

$$\frac{1}{\pi}\int_0^\infty s^{-\frac{1}{2}}E_s(r)\,ds = -H(1,k)^{-1}r^{1-k}. \tag{4}$$

D. The distribution spaces in E_k. The use of Banach space methods in partial differential equations leads naturally to the consideration of spaces of functions whose derivatives are functions in L^p, and to the duals of these spaces.

Definition 2. f is in $L_1{}^p(E_k)$ if f is in $L^p(E_k)$ and, for every $i = 1, \cdots, k$ there is a function f_i in L^p such that for every ϕ in C_∞ with compact support $-(f, \partial\phi/\partial x_i) = (f_i, \phi)$; or (if $p > 1$), $|(f, \partial\phi/\partial x_i)| \leqq A \|\phi\|_q$, $q^{-1} + p^{-1} = 1$. Then $f_i = (\partial/\partial x_i)f$. f is in $L_m{}^p$ $(m > 0)$ if f is in $L_1{}^p$ and all first order derivatives of f are in $L_{m-1}{}^p$.

There is a topology in $L_m{}^p$ defined by the norm

$$\|f\|_m = \Big(\sum_{0 \leqq |\alpha| \leqq m} \|(\partial/\partial x)^\alpha f\|_p{}^p\Big)^{1/p}.$$

The distribution space $L_{-m}{}^p$ $(m \geqq 0, 1 < p < \infty)$ is the space of bounded linear functionals on $L_m{}^q$, where $p^{-1} + q^{-1} = 1$. $L_0{}^q = L^q$, $(1 < q < \infty)$.

It is clear that $L_m{}^p \subset L_{m-1}{}^p$, $m = 0, \pm 1, \pm 2, \cdots$. Functions in C_∞ with compact support are dense in $L_m{}^p$ for $m \geqq 0$. (See e.g. [2], Lemma 1.) For $m \geqq 0$, $L_m{}^p$ is closed under multiplication by bounded functions ϕ in C_m, with bounded derivatives, and $\|\phi f\|_m \leqq C(m)\sup|(\partial/\partial x)^\alpha\phi|\,\|f\|_m$ for $0 \leqq |\alpha| \leqq m$. The same is true for $m < 0$: if h is in $L_m{}^p$, then ϕh is the

linear functional on L_{-m}^q defined by $(\phi h, f) = (h, \phi^* f)$, where f is in L_{-m}^q, and (a, b) denotes the value of the functional a on the element b. $h_1 = h_2$ on the compact set C if $\phi h_1 = \phi h_2$ for every ϕ in C_m with support in C.

These spaces play a central role in this paper; the following facts about them will be used frequently.

Remark (a). If H is a C_β^∞ singular integral operator, then H and H^* are bounded on L_m^p for $m \leqq [\beta]$; $(\partial/\partial x_\nu)H - H(\partial/\partial x_\nu)$ and $(\partial/\partial x_\nu)H^* - H^*(\partial/\partial x_\nu)$ are bounded on L_m^p for $m < [\beta]$.

Proof. Write $H = \sum a_{ij} T_{ij}$. Since T_{ij} commutes with differentiation, its norm on L_m^p is bounded by that on L^p. Since a_{ij} is in C_β it is also bounded on L_m^p for $m \leqq [\beta]$, and its norm is $\leqq C(m) \sup_{0 \leqq |\alpha| \leqq m} |(\partial/\partial x)^\alpha a_{ij}|$. Since the supremum on the right is $O(i^{-r})$ for every r, $\| T_{ij} \|$ has a bound independent of i and j, and the number of T_{ij} for fixed i is $O(i^{k-2})$, the series for H can be summed to get $\| H \|_{m,m} < \infty$, where $\| \ \|_{m,m}$ is the norm of an operator from L_m^p to L_m^p. The same method applies to $H^* = \sum T_{ij} a_{ij}^*$,

$$(\partial/\partial x_\nu)H - H(\partial/\partial x_\nu) = \sum (\partial a_{ij}/\partial x_\nu) T_{ij}, \text{ and}$$

$$(\partial/\partial x_\nu)H^* - H^*(\partial/\partial x_\nu) = \sum T_{ij} (\partial a_{ij}/\partial x_\nu)^*.$$

Remark (b). Let $\| K \|_{j,m}$ denote the norm of K as an operator from L_j^p to L_m^p. Suppose $\| K \|_{0,1} < C$ and $\| (\partial/\partial x_i)K - K(\partial/\partial x_i) \|_{m,m} < C$ for $m < n$, $i = 1, \cdots, k$. Then $\| K \|_{m,m+1} < A(m)C$ for $m < n$; all the C's are the same, and $A(m)$ depends only on m and p.

The proof is by induction on m. If f is in L_m^p, $1 \leqq m < n$,

$$\| (\partial/\partial x_i)Kf \|_m \leqq \| K \partial f/\partial x_i \|_m + \| ((\partial/\partial x_i)K - K(\partial/\partial x_i))f \|_m$$

$$\leqq CA(m-1) \| \partial f/\partial x_i \|_{m-1} + C \| f \|_m \leqq CA(m-1) \| f \|_m + C \| f \|_m,$$

so $\| Kf \|_{m+1} \leqq CA(m) \| f \|_m$.

Remark (c). Let T be an operator on L^p for which $(\partial/\partial x_i)T$ and $T(\partial/\partial x_i)$ have some meaning. Then let $C_i(T) = (\partial/\partial x_i)T - T(\partial/\partial x_i)$. If $C_i(T)$ can be similarly combined with differentiation, let $C_j C_i(T) = C_j(C_i(T)) = C_i(C_j(T))$, and generally $C_\alpha(T) = C_1^{\alpha_1} \cdots C_k^{\alpha_k}(T)$. Then if $\| C_\alpha(T) \|_{0,0} < C$ for $|\alpha| < n$, $\| T \|_{m,m} < CA(m)$ for $m < n$.

To see this, let $(\partial/\partial x)^\alpha = (\partial/\partial x_{i_1}) \cdots (\partial/\partial x_{i_m}) = D_1 \cdots D_m$, and $C^\nu(T) = C_{i_\nu}(T)$. Then it can be shown by induction that

$$D_1 \cdots D_m T = T D_1 \cdots D_m + \sum_{\nu=1}^{m} C^\nu(T) D_1 \cdots \hat{D}_\nu \cdots D_m$$

$$+ \sum_{1 \leqq \nu < \mu \leqq m} C^\nu C^\mu(T) D_1 \cdots \hat{D}_\nu \cdots \hat{D}_\mu \cdots D_m + \cdots + C_1 C_2 \cdots C_m(T);$$

here $D_1 \cdots \check{D}_\nu \cdots \check{D}_\mu \cdots D_m$ indicates the product of all the D_i except D_ν and D_μ. Thus if we assume that f is in L_m^p,

$$D_1 \cdots D_m T f = T D_1 \cdots D_m f + \cdots + C_1 C_2 \cdots C_m(T) f$$

is in L^p, and $\| Tf \|_m \leqq CA(m) \| f \|_m$.

LEMMA 2. *Let $K(x,y)$ be continuously differentiable for $x \neq y$; $|K(x,y)| < \phi(x-y)$ with $\int \phi(z)\,dz < C$ and $\phi(x) = O(|x|^{1-k})$ as $|x| \to 0$; and let*

$$\lim_{\epsilon \to 0} \epsilon^{k-1} \int_{|x-y|=\epsilon} K(y,x)\gamma_i\, d\sigma_y = -\lim_{\epsilon \to 0} \epsilon^{k-1} \int_{|x-y|=\epsilon} K(x,y)\gamma_i\, d\sigma_y = k_i(x),$$

where $\gamma_i = (y_i - x_i)/|y-x|$ and the first limit exists uniformly in x, and $|k_i| < C$; and if $H(x,y)$ stands for any of the kernels $\partial K(x,y)/\partial y_i$, $\partial K(x,y)/\partial x_i$, $\partial K(y,x)/\partial y_i$, $\partial K(y,x)/\partial x_i$, let $\lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} H(x,y) f(y)\,dy$ exist uniformly in x for f in C_1 with compact support, and converge strongly to a bounded operator on L^p. Then

(i) $$\int K(x,y)\,(\partial f/\partial y_i)\,dy = -\lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} (\partial K(x,y)/\partial y_i) f(y)\,dy + k_i(x) f(x), \quad f \text{ in } L_1^p$$

(ii) $$(\partial/\partial x_i) \int K(x,y) f(y)\,dy = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} (\partial K(x,y)/\partial x_i) f(y)\,dy + k_i(x) f(x), \quad f \text{ in } L^p$$

(iii) $$\int K(x,y)\,(\partial f/\partial x_i)\,dx = -\lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} (\partial K(x,y)/\partial x_i) f(x)\,dx - k_i(y) f(y), \quad f \text{ in } L_1^p$$

(iv) $$(\partial/\partial y_i) \int K(x,y) f(x)\,dx = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} (\partial K(x,y)/\partial y_i) f(x)\,dx - k_i(y) f(y), \quad f \text{ in } L^p.$$

Proof. (i) Let f be in C_1 and have compact support. Then

$$\begin{aligned}
&\lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} K(x,y)\,(\partial f/\partial y_i)\,dy \\
&\quad = -\lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} (\partial K(x,y)/\partial y_i) f(y)\,dy - \lim_{\epsilon \to 0} \int_{|x-y|=\epsilon} \epsilon^{k-1} K(x,y) f(y) \gamma_i\, d\sigma_y \\
&\quad = -\lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} (\partial K/\partial y_i) f(y)\,dy + k_i(x) f(x).
\end{aligned}$$

The result then holds for f in $L_1{}^p$ by approximation.

(ii) Let g and f be in C_1 and have compact support. Then

$$-(\partial g/\partial x_i, \int K(x,y)f(y)\,dy) = -\int f^*(y) \int (\partial g/\partial x_i)K^*(x,y)\,dx\,dy$$

$$= \int f^*(y)\{\lim_{\epsilon\to 0}\int_{|x-y|>\epsilon} g(x)\,(\partial K^*/\partial x_i)\,dx + k_i{}^*(y)g(y)\}dy \quad \text{(by (i))}$$

$$= \lim_{\epsilon\to 0}\int\int_{|x-y|>\epsilon} f^*(y)g(x)\,(\partial K^*/\partial x_i)\,dy\,dx + \int k_i{}^*f^*g(y)\,dy$$

$$= \int g(x)\{\lim_{\epsilon\to 0}\int_{|x-y|>\epsilon} f^*(y)\,(\partial K^*/\partial x_i)\,dy + k_i{}^*(x)f^*(x)\}dx.$$

Hence if f is in C_1 with compact support, $\int K(x,y)f(y)\,dy$ has the (weak) derivative given in (ii); the result holds again by approximation for f in L^p.

(iii) and (iv) follow from (i) and (ii) respectively.

We introduce as a notational convenience $(\partial/\partial x_i + \partial/\partial y_i)f(x,y) = \partial f/\partial x_i + \partial f/\partial y_i$, and $(\partial/\partial x + \partial/\partial y)^\alpha f(x,y) = \prod(\partial/\partial x_i + \partial/\partial y_i)^{\alpha_i} f(x,y)$.

LEMMA 3. *If $K(x,y)$ has continuous derivatives up to and including order n for $y \neq x$, and $(\partial/\partial x + \partial/\partial y)^\alpha K(x,y)$ satisfies for $|\alpha| < n$ all the conditions imposed on K in Lemma 2, and for every $i = 1, \cdots, k$, then $\| K \|_{m,m+1} < A(n,p)B$ and $\| K^* \|_{m,m+1} < A(n,p)B$ for $m < n$; B is an upper bound for the C of Lemma 2, $\| K \|_{0,1}$ and $\| K^* \|_{0,1}$.*

Proof. Let $(\partial/\partial x_j)K - K(\partial/\partial x_j) = K_j$; by Lemma 2, K_j is the operator with kernel $(\partial/\partial x_j + \partial/\partial y_j)K(x,y) = K_j(x,y)$. Then $(\partial/\partial x + \partial/\partial y)^\alpha K_j(x,y)$ satisfies for $|\alpha| < n-1$ the conditions on K in Lemma 2. An easy induction then shows that for $|\alpha| < n$, $C_\alpha(K_j)$ is the operator with kernel

$$(\partial/\partial x + \partial/\partial y)^\alpha K_j(x,y), \text{ and } \| C_\alpha(K_j) \|_{0,0} < B$$

By Remark (c), $\| K \|_{m,m} < BA(m)$ for $m < n$. By Lemma 2, $\| K \|_{0,1} < B$, so that Remark (b) gives $\| K \|_{m,m+1} < A(n,p)B$ for $m < n$.

Since the conditions on the kernel of K^* are the same as those on K, $\| K^* \|_{m,m+1} < A(n,p)B$ for $m < n$.

COROLLARY. *Let $H(x,z)$ be in $C_\beta{}^\infty$ and homogeneous of degree $1-k$ in z; $\theta(x,y)$ be in C_β on E_{2k} with $|\theta(x,y)| < \phi(x-y)$, where ϕ is bounded and $\phi(x) = 0$ for $|x| > R$. Then if K is the operator with kernel*

$$\theta(x,y)H(x,x-y), \text{ we have } \| K \|_{m,m+1} < A \text{ and } \| K^* \|_{m,m+1} < A$$

for $m < [\beta]$.

Proof. Let $H_\alpha(x,z) = (\partial/\partial x)^\alpha H(x,z)$. Then

$$
\begin{aligned}
&(\partial/\partial x + \partial/\partial y)^\alpha \theta(x,y) H(x, x-y) \\
&\quad = (\partial/\partial\xi)^\alpha \theta(\tfrac{1}{2}(\xi+\eta), \tfrac{1}{2}(\xi-\eta)) H(\tfrac{1}{2}(\xi+\eta), \eta) \\
&\quad = \sum_{\gamma \leqq \alpha} \{\alpha!/\gamma!(\alpha-\gamma)!\} (\partial/\partial\xi)^\gamma \theta(\tfrac{1}{2}(\xi+\eta), \tfrac{1}{2}(\xi-\eta)) (\partial/\partial\xi)^{\alpha-\gamma} \\
&\qquad\qquad \times H(\tfrac{1}{2}(\xi+\eta), \eta) \\
&\quad = \sum_{\gamma \leqq \alpha} \{\alpha!/\gamma!(\alpha-\gamma)!\} (\partial/\partial x + \partial/\partial y)^\gamma \theta(x,y)\, 2^{-|\alpha-\gamma|} H_{\alpha-\gamma}(x, x-y).
\end{aligned}
$$

Hence we need only show that if $\theta(x,y)$ is in C_1, $|\theta(x,y)| < \phi(x-y)$ with ϕ bounded and of compact support, and if $H(x,z)$ is in C_1^∞ and homogeneous of degree $1-k$ in z, then θH satisfies the conditions of Lemma 2. The first one is obvious. For the second,

$$
\begin{aligned}
&-\lim_{\epsilon\to 0} \epsilon^{k-1} \int_{|x-y|=\epsilon} \theta(x,y) H(x, x-y) ((y_i - x_i)/|y-x|)\, d\sigma_y \\
&\quad = \lim_{\epsilon\to 0} \epsilon^{k-1} \int_{|z|=\epsilon} \theta(x, x-z) H(x,z) (z_i/|z|)\, d\sigma_z = \theta(x,x) \int_{|z|=1} H(x,z) z_i\, d\sigma_z.
\end{aligned}
$$

In the same way,

$$
\begin{aligned}
&\lim_{\epsilon\to 0} \epsilon^{k-1} \int_{|x-y|=\epsilon} \theta(y,x) H(y, y-x) ((y_i - x_i)/|y-x|)\, d\sigma_y \\
&\qquad\qquad = \theta(x,x) \int_{|z|=1} H(x,z) z_i\, d\sigma_z.
\end{aligned}
$$

For the last condition we take as an example the kernel

$$(\partial/\partial x_i)\{\theta(x,y) H(x, x-y)\}.$$

Let

$$H_i(x,z) = \partial H(x,z)/\partial x_i, \quad H^i(x,z) = \partial H(x,z)/\partial z_i, \quad \text{and} \quad \theta_i(x,y) = \partial\theta(x,y)/\partial x_i.$$

Then the limit that concerns us is

$$
\begin{aligned}
&\lim_{\epsilon\to 0} \int_{|x-y|>\epsilon} \{\theta_i(x,y) H(x, x-y) + \theta(x,y) H_i(x, x-y)\} f(y)\, dy \\
&\qquad + \lim_{\epsilon\to 0} \int_{\epsilon<|x-y|<R} \{\theta(x,y) - \theta(x,x)\} H^i(x, x-y) f(y)\, dy \\
&\qquad + \lim_{\epsilon\to 0} \int_{R>|x-y|>\epsilon} \theta(x,x) H^i(x, x-y) f(y)\, dy.
\end{aligned}
$$

The kernel in each of the first two integrals is dominated by $\psi(x-y)$ for some ψ in L^1, so that they converge uniformly for bounded f, and define bounded operators on L^p. The last operator is a truncated C_β^∞ singular

integral operator, by Lemma 1, and it follows from the results in [1] that this is a bounded operator on L^p. It converges uniformly for f in C_1 since

$$\lim_{\epsilon\to 0}\int_{R>|x-y|>\epsilon}\theta(x,x)H^i(x,x-y)f(y)\,dy$$
$$=\lim_{\epsilon\to 0}\int_{R>|x-y|>\epsilon}\theta(x,x)H^i(x,x-y)\{f(y)-f(x)\}\,dy;$$

this integrand is dominated by a $\psi(x-y)$ with ψ in L^1.

PROPOSITION. *$L_m{}^p$ is topologically isomorphic to $L_{m+1}{}^p$, $1<p<\infty$.*

Proof. The isomorphism is defined for $m\geqq 0$ by

$$G(f)=(I-i\Lambda)(I-\Delta)^{-1}f,$$

where Λ is as in section B above. We will prove that G maps $L_m{}^p$ continuously into $L_{m+1}{}^p$; one sees from the Fourier transforms that $(I+i\Lambda)G$ and $G(I+i\Lambda)$ are the identity operators on $L_m{}^p$ and $L_{m+1}{}^p$ respectively. Since Λ and the Riesz transform R_j commute with differentiation, and R_j is bounded on $L_m{}^p$, $(m\geqq 0)$, Λ maps $L_m{}^p$ continuously into $L_{m-1}{}^p$ for $m>0$, so that G^{-1} is bounded from $L_{m+1}{}^p$ to $L_m{}^p$.

We show that G is defined and continuous from $L_m{}^p$ into $L_{m+1}{}^p$ by proving that the Green's function of $\Delta-1$ maps $L_m{}^p$ into $L_{m+2}{}^p$. The Green's function is, from (1), $E_1(|x-y|)=B|x-y|^{1-\frac{1}{2}k}K_{\frac{1}{2}k-1}(|x-y|)$. Let f be in L^p. Then by Lemma 2

$$(\partial/\partial x_i)(\Delta-1)^{-1}f=\lim_{\epsilon\to 0}\int_{|x-y|>\epsilon}(\partial E_1/\partial x_i)f(y)\,dy$$
$$-f(x)\lim_{\epsilon\to 0}\epsilon^{k-1}\int_{|x-y|=\epsilon}E_1(|x-y|)\gamma_i\,d\sigma_y=\int(\partial E_1/\partial x_i)f(y)\,dy,$$

since $E_1(r)$ is $O(r^{2-k})$ and $E_1'(r)=O(r^{1-k})$. Then again

$$(\partial^2/\partial x_i\partial x_j)(\Delta-1)^{-1}f=\lim_{\epsilon\to 0}\int_{|x-y|>\epsilon}(\partial^2E_1/\partial x_i\partial x_j)f(y)\,dy$$
$$-f(x)\lim_{\epsilon\to 0}\int_{|x-y|=\epsilon}(\partial E_1/\partial x_i)\gamma_j\epsilon^{k-1}\,d\sigma_y.$$

Since $\partial|x-y|/\partial x_i=-(y_i-x_i)/|y-x|=-\gamma_i$, (2) shows that this last limit is $(\delta_{ij}/k)f(x)$. To show $\partial^2E_1/\partial x_i\partial x_j$ is summable as required in Lemma 2, write it as

$$X(|x-y|)A\partial^2|x-y|^{2-k}/\partial x_i\partial x_j$$
$$+X(|x-y|)\partial^2\{E_1(|x-y|)--A|x-y|^{2-k}\}/\partial x_i\partial x_j$$
$$+\{1-X(|x-y|)\}\partial^2E_1/\partial x_i\partial x_j,$$

where $X(t)=1$ if $t<1$ and $X(t)=0$ if $t>1$. That the first term is summable is proved in [1], and the last two are in L^1, according to the estimates (2) and (3).

This shows that $(\Delta-1)^{-1}$ is bounded from L^p to $L_2{}^p$. Since it commutes with differentiation, it is also bounded from $L_m{}^p$ to $L_{m+2}{}^p$.

This argument is not exactly right for $k=2$, but the modifications are trivial.

Now that the proposition is proved for $m \geqq 0$, it is seen to be true for all m. A consequence of this is that $L_m{}^p$ is isomorphic to L^p, and hence reflexive, for $\infty > p > 1$.

Operators which add to the number of derivatives of a function in L^p are in some sense "smoothing" operators. This can be made precise.

Definition 3. T is smoothing of order n if T is a bounded transformation of $L_m{}^p$ into $L_{m+1}{}^p$ for $-n \leqq m < n$. Equivalently, T is defined on L^p and bounded from $L_m{}^p$ to $L_{m+1}{}^p$ $(0 \leqq m < n)$; and T^* is bounded from $L_m{}^q$ to $L_{m+1}{}^q$ $(0 \leqq m < n)$.

For example, the operator $(I-i\Lambda)(I-\Delta)^{-1}$ of the previous proposition is smoothing of order n for every n.

II. Singular Integral Operators on a Compact Manifold.

Throughout this section, unless otherwise specified, M will denote a compact manifold of dimension k, class C_n, $(n \geqq 3)$. The singular integral operators will be characterized by their local behavior. The conventions of tensor calculus will generally not be used.

A. The functional spaces on M. We can define $L_r{}^p(M)$ $(1 < p < \infty$, $r \leqq n)$ as a topological vector space by using a partition of the identity on M in the form of a finite collection $\{\phi_i\}$ of functions in C_n satisfying

i) $\phi_i \geqq 0$, $\sum \phi_i = 1$;

ii) the support of ϕ_i lies in the domain of a single coordinate system.

With each ϕ_i we can associate a particular coordinate system x^i. Then if f is a function defined almost everywhere in each coordinate domain, $\phi_i f$ can be considered as a function of compact support in E_k. We say that f is in $L_r{}^p(M)$ if $\phi_i f$ is in $L_r{}^p(E_k)$ for each i. If $\| \ \|$ denotes the norm on $L^p(E_k)$, then we define a topology on $L_r{}^p(M)$ by the norm

$$\| f \|_{r,p} = \Big(\sum_{|\alpha| \leqq r} \sum_i \| \phi_i{}^{1/p} (\partial/\partial x_i)^\alpha f \|^p \Big)^{1/p}.$$

When $r=0$, this is $L^p(M,v)$, where v is the measure $\sum \phi_i dx^i$. The identification of functionals on L^p as functions in L^q can be made using the inner product given by this measure.

It should be checked that the norms obtained with different coordinate systems and different $\{\phi_i\}$ are all equivalent. Suppose first that $x(\xi)$ is a coordinate change in U_i, the support of ϕ_i. Then if f is in $L_m{}^p$ (in the ξ system), and $|\alpha| \leqq m$, $(\partial/\partial x)^\alpha f = \sum_{|\beta| \leqq |\alpha|} c_\beta (\partial/\partial \xi)^\beta f$, where the c_β are bounded functions. Thus the norm from the x system is dominated by that of the ξ system; and they are seen to be equivalent by reversing the roles of x and ξ. To show independence of the choice of $\{\phi_i\}$, we need only consider the case where the second system is of the form $\{\phi_{ij}\}$ with $\sum_j \phi_{ij} = \phi_i$, and using a single coordinate system in U_i. Then

$$\left(\sum_{i,j,\alpha} \| \phi_{ij}{}^{1/p} (\partial/\partial x)^\alpha f \|^p\right)^{1/p} = \left(\sum_{i,\alpha} \| \phi_i{}^{1/p} (\partial/\partial x)^\alpha f \|^p\right)^{1/p}.$$

The distribution space $L_{-r}{}^p(M)$ is defined, as in the Euclidean case, to be the dual of $L_r{}^p$. It will be a corollary of some later results that these spaces are isomorphic to $L^p(M)$, hence reflexive, for $1 < p < \infty$.

A smoothing operator of order m is an operator T on $L^p(M)$ such that T maps $L_r{}^p$ continuously into $L_{r+1}{}^p$ and T^* maps $L_r{}^q$ continuously into $L_{r+1}{}^q$ for $0 \leqq r < m$. T is such an operator if and only if, for every ϕ and ψ in C_n with support in a single coordinate neighborhood, $\phi T \psi$, considered as an operator on $L^p(E_k)$, is smoothing of order m.

B. Singular integral operators on M and their symbols.

Definition 4. An operator T defined on $L^p(M)$ is of type $C_\beta{}^\infty$ $(\beta \leqq n-1)$ if it satisfies

i) for each ϕ and ψ in C_n on M with disjoint (compact) support, $\phi T \psi$ is completely continuous and smoothing of order $[\beta]$ on all $L^p(M)$, $1 < p < \infty$;

ii) for each ϕ and ψ in C_n with support in a common coordinate domain with coordinates x, $\phi T \psi = \phi H \psi + \mathrm{R}$, where H is a Euclidean $C_\beta{}^\infty$ singular integral operator

$$Hf(x) = a(x) f(x) + \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} h(x, x-y) f(y)\, dy,$$

and R is completely continuous and smoothing of order $[\beta]$ on $L^p(M)$, $1 < p < \infty$.

Definition 5. The symbol of a C_β^∞ operator T on M is the function on the cotangent bundle of M defined by

$$\sigma(T)(p, \sum \xi_i\, dx_i) = a(x(p)) + \lim_{\epsilon \to 0} \int_{\epsilon^{-1} > |\eta| > \epsilon} \exp(i \sum \xi_j \eta_j) h(x(p), \eta)\, d\eta$$

where $a(x)$ and $h(x, \eta)$ are the functions in Definition 4, $\eta = (\eta_1, \cdots, \eta_k)$, $|\eta|^2 = \sum \eta_i^2$, and p is in the common support of ϕ and ψ.

The existence of such operators will be established in Theorem 2 below, and the independence of symbols and coordinate systems in Theorem 1.

Spherical harmonics provide a useful link between the operator T and its symbol $\sigma(T)$. If $h(x, z) = \sum a_{nm}(x) Y_{nm}(z) |z|^{-k}$, the Euclidean symbol of the operator $H = a + \int h$ is

$$a(x) + \lim_{\epsilon \to 0} \int_{\epsilon^{-1} > |\zeta| > \epsilon} \exp(i \sum z_j \zeta_j) h(x, \zeta)\, d\zeta = a(x) + \sum a_{nm}(x) \gamma_n Y_{nm}(z),$$

wherein $\gamma_n = i^n \pi^{k/2} \Gamma(n/2) / \Gamma(\frac{1}{2}\{n + k\})$. For this see [2].

If H is the operator of Definition 4, then

$$\sigma(T)(p, \sum \xi_i\, dx_i) = a(x(p)) + \sum a_{nm}(x(p)) \gamma_n Y_{nm}(\xi),$$

where ξ is the point in E_k with coordinates $(\xi_1, \cdots, \xi_k)$. Thus the symbol of a C_β^∞ operator is a function on the cotangent bundle which is homogeneous of degree zero on each cotangent space, and in C_∞ with respect to cosphere variables; and each derivative with respect to these variables is in C_β on the cosphere bundle. Such a function will be called a C_β^∞ function on the cosphere bundle $CS(M)$.

LEMMA 4. *If $\phi T\psi - \phi H_1 \psi$ and $\phi T\psi - \phi H_2 \psi$ are both completely continuous, then $\sigma(H_1)(x(p), z) = \sigma(H_2)(x(p), z)$ if p is in the common support of ϕ and ψ.*

THEOREM 1. *The symbol $\sigma(T)$ is independent of the coordinates used to define it.*

Proof of Lemma 4. By our hypotheses, $\phi(H_1 - H_2)\psi$ is a completely continuous operator on $L^p(E_k)$. Let U be the common support of ϕ and ψ (considered as a subset of E_k). Let $L^p(U)$ be the functions in $L^p(E_k)$ which vanish off U. Consider the vector space C of symbols of (Euclidean) C_0^∞ operators with the property that $\sigma(H)(x, z)$ vanishes off U, and H is completely continuous on $L^p(U)$. We will see that $\sigma\{\phi\psi(H_1 - H_2)\}$ is in C, and that C is trivial.

First, let $a(x)$ and $b(x)$ have compact support, $a(x)$ in C_0 and $b(x)$ in C_1;

and let $h(x,z)$ be a C_0^∞ kernel. Then $a(x)\{b(x)-b(y)\}h(x,x-y)$ is uniformly $O(|x-y|^{1-k})$, hence defines a completely continuous operator $a(bH-Hb)$ on $L^p(U)$. But any b in C_0 can be approximated uniformly by functions in C_1, so that $a(bH-Hb)$ is approximated in norm by completely continuous operators if $a(x)$ and $b(x)$ are only continuous. Thus $\phi\psi(H_1-H_2) = \phi(H_1-H_2)\psi + \phi\{\psi(H_1-H_2)-(H_1-H_2)\psi\}$ has a symbol in C.

Now let Y_{nm} be a normalized spherical harmonic, and

$$T_{nm}f(x) = \lim_{\epsilon\to 0}\int_{|x-y|>\epsilon} Y_{nm}(x-y)f(y)\,dy.$$

Then if $\sigma(H)$ is in C, K is a C_0^∞ operator, $H=\sum a_{nm}T_{nm}$, $K=\sum b_{nm}T_{nm}$ (the b_{nm} having support in U) and $H\circ K$ is the C_0^∞ operator with symbol $\sigma(H)\sigma(K)$, we have $H\circ K = HK - \sum a_{nm}(T_{nm}b_{\nu\mu}-b_{\nu\mu}T_{nm})T_{\nu\mu}$. Then as in the previous paragraph $a_{nm}(T_{nm}b_{\nu\mu}-b_{\nu\mu}T_{nm})$ is completely continuous on $L^p(U)$, so that $H\circ K$ is also. Thus C is closed under multiplication by C_0^∞ functions $F(x,z)$, homogeneous of degree zero in z, vanishing for x not in U.

Now suppose $F(x_0,z)$ is not identically zero in z, but $F(x,z)$ is in C. Let $\theta(x_0)=1$, and θ vanish outside a spherical neighborhood of x_0 in U, and $F'(x,z)=\theta(x)F(x,z)$. Let u be a rotation of E_k about x_0, and $F_u(x,z)=F'(u(x),u(x_0+z)-x_0)$. Then F_u is also clearly in C; and we can find finitely many u_i so that $G(x,z)=\sum|F_{u_i}(x,z)|^2$ is bounded away from zero in a neighborhood of x_0. Let $a(x)$ be continuous and have support in this neighborhood. Then $a(x)G(x,z)^{-1}z_j|z|^{-1}$ is in C_0^∞ and $a(x)G(x,z)^{-1}z_j|z|^{-1}G(x,z)=a(x)z_j|z|^{-1}$ is in C. But $z_j|z|^{-1}$ is the symbol of the Riesz transform R'_j and so aR_j is completely continuous on $L^p(U)$. Then $a^2I=\sum(aR_j)^2+\sum a(aR_j-R_ja)R_j$ is completely continuous. This contradiction establishes Lemma 4.

As a consequence, the H appearing in part (ii) of Definition 4 is essentially unique. We need two more results to establish Theorem 1.

LEMMA 5. *Let $h(x,z)$ be a C_β^∞ function, homogeneous of degree $-k$, $\beta\geqq 0$, and $\xi(x)$ a C_n change of coordinates. Let $\rho^2(x,y)=\sum\{\xi_i(x)-\xi_i(y)\}^2$,*

$$H_\epsilon f(x)=\int_{|x-y|>\epsilon} h(x,x-y)f(y)\,dy, \text{ and } H'_\epsilon f(x)=\int_{\rho(x,y)>\epsilon} h(x,x-y)f(y)\,dy.$$

f in $L^p(E_k)$. Then $H_\epsilon f$ and $H'_\epsilon f$ have limits Hf and $H'f$ respectively, as $\epsilon\to 0$, and $H'f(x)-Hf(x)=D(x)f(x)$, where

$$D(x)=-\int_{|z|=1} h(x,z)\log b(x,z)\,d\sigma \text{ and } b(x,z)=\lim_{t\to 0} t/\rho(x,x+tz).$$

$D(x)$ is in $C_{n-1}\cap C_\beta$.

Proof. As $t \to 0$,

$$b^{-2}(x,z) = \lim \rho^2(x, x+tz)/t^2 = \lim \sum (\{\xi_i(x+tz) - \xi_i(x)\}/t)^2$$
$$= \sum_i \{\sum_j (\partial \xi_i/\partial x_j)(x) z_j\}^2,$$

so $b(x,z)$ is in C_{n-1}. Now for f in C_1 with compact support

$$\text{i)} \qquad H_\epsilon' f(x) = \int_{\substack{\rho(x,y)>\epsilon \\ |x-y|\leqq 1}} h(x, x-y)\{f(y) - f(x)\}dy$$
$$+ f(x) \int_{\substack{\rho(x,y)>\epsilon \\ |x-y|\leqq 1}} h(x, x-y)\,dy + \int_{|x-y|>1} h(x, x-y) f(y)\,dy.$$

The first term is convergent and the limit of the first and last terms together is $\lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} h(x, x-y) f(y)\,dy = Hf(x)$. To calculate the second, let $b(\xi, x, z)$ be defined for $|z| = 1$ and small ϵ by $\rho(x, x + zb(\epsilon, x, z)) = \epsilon$; and let $m(\epsilon, x) = \max_{|z|=1} b(\epsilon, x, z)$. Then

$$\int_{\rho(x,y)>\epsilon} h(x, x-y)\,dy = \int_{|z|=1} \int_{b(\epsilon,x,z)}^{m(\epsilon,x)} h(x, tz)\, t^{k-1}\, dt\, d\sigma$$
$$= \int_{|z|=1} h(x,z) \log\{m(\epsilon, x)/b(\epsilon, x, z)\}\, d\sigma.$$

Now

$$\lim_{\epsilon \to 0} b(\epsilon, x, z)/\epsilon = \lim_{b \to 0} b/\rho(x, x+zb) = b(x,z),$$

so

$$\lim m(\epsilon, x)/b(\epsilon, x, z) = \{\max_{|z|=1} b(x,z)\}/b(x,z),$$

and

$$\lim_{\epsilon \to 0} \int_{\rho(x,y)>\epsilon} h(x, x-y)\,dy = -\int_{|z|=1} h(x,z) \log b(x,z)\, d\sigma = D(x).$$

Hence (i) approaches the desired expression as $\epsilon \to 0$, pointwise and in L^p norm. The same result holds also for f in L^p by approximation, and Lemma 3 is proved.

Note that $b(x,z)$ is even in z, so that if $h(x,z)$ is odd in z, the method of summation is irrelevant.

LEMMA 6. *Let $x(\xi)$ be a C_n change of coordinates $(n \geqq 3)$ in a neighborhood U of E_k. Let $x(\xi) = x$, $x(\eta) = y$, $a_{ij} = (\partial x_i/\partial \xi_j)$, $A = (a_{ij})$. Let $h(x,z)$ be in C_β^∞, $0 \leqq \beta \leqq n-1$, and homogeneous of degree $-k$. Then $h(x, x-y) = h(x, A(\xi - \eta)) + R(\xi, \eta)$, where $R(\xi, \eta)$ is uniformly*

$O(|\xi-\eta|^{1-k})$. *If* $1 \leqq \beta \leqq n-1$, *then* $\phi(\xi)\psi(\eta)R(\xi,\eta)$ *defines an operator which is smoothing of order* $[\beta]$ *on* $L^p(U)$ *for every* ϕ *and* ψ *in* C_{n-1} *with support in* U.

Proof. By Taylor's expansion,

$$\begin{aligned} x_i(\eta) &= \sum_{0 \leqq |\alpha| < 3} x_i^{(\alpha)}(\xi)\,(1/\alpha!)\,(\eta-\xi)^\alpha - P_i(\xi,\eta) \\ &= x_i(\xi) + \sum_j a_{ij}(\eta_j - \xi_j) - Q_i(\xi,\eta) - P_i(\xi,\eta). \end{aligned}$$

$P_i(\xi,\eta) = O(|\xi-\eta|^2)$ and any first order derivative of $P_i(\xi,\eta)$ is $O(|\xi-\eta|^2)$. Now let

$$Q = \{Q_1(\xi,\eta), \cdots, Q_k(\xi,\eta)\}, \qquad P = \{P_1(\xi,\eta), \cdots, P_k(\xi,\eta)\},$$
$$\text{and } h^{(\alpha)}(x,z) = (\partial/\partial z)^\alpha h(x,z).$$

Then

$$\begin{aligned} h(x,x-y) &= h(x, A(\xi-\eta) + Q + P) \\ &= h(x, A(\xi-\eta)) + \sum_{|\alpha|=1} Q^\alpha h^{(\alpha)}(x, A(\xi-\eta)) + \sum_{|\alpha|=1} P^\alpha h^{(\alpha)}(x, A(\xi-\eta)) \\ &\quad + |A(\xi-\eta)|^{-k-2} \sum_{|\alpha|=2} (Q+P)^\alpha \int_0^1 (1-t) h^{(\alpha)}(x, |A(\xi-\eta)|^{-1} \\ &\qquad \times \{A(\xi-\eta) + t(Q+P)\})\,dt \\ &= h(x, A(\xi-\eta)) + h_1(\xi, \xi-\eta) + R_1(\xi,\eta) + R_2(\xi,\eta). \end{aligned}$$

Here $h_1(\xi,z)$ is homogeneous of degree $1-k$, R_1 and R_2 are uniformly $O(|\xi-\eta|^{2-k})$, and any first derivative of R_1 or R_2 is uniformly $O(|\xi-\eta|^{1-k})$.

Then by Lemma 2 and the Corollary of Lemma 3,

$$\phi(x)\{h_1(\xi,\xi-\eta) + R_1(\xi,\eta) + R_2(\xi,\eta)\}\psi(y) = \phi R\psi$$

is a bounded operator from L^p to L_1^p. To show boundedness from L_m^p to L_{m+1}^p for $m < [\beta]$, note that $\phi R\psi$ measures the difference between

$$\lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \phi(x) h(x, x-y)\psi(y) f(y)\,dy \text{ and}$$

$$\lim_{\epsilon \to 0} \int_{|\xi-\eta|>\epsilon} \phi(x(\xi)) h_0(\xi, \xi-\eta)\psi(x(\eta)) f(\eta) \det(\partial x_i/\partial \eta_j)\,d\eta$$
$$+ \phi(x(\xi))\psi(x(\xi)) D(\xi) f(\xi),$$

where $D(\xi)$ is in C_β as in Lemma 5. It is a trivial modification of Remark (a) that the difference between these two operators, $\phi R\psi$, satisfies $\|(\partial/\partial x_i)\phi R\psi - \phi R\psi(\partial/\partial x_i)\|_{m,m} < C$, and the same for the adjoint; then by Remark (b), $\|\phi R\psi\|_{m,m+1} < C$ and $\|\psi^* R^* \phi^*\|_{m,m+1} < C$ for $m < [\beta]$.

Proof of Theorem 1. By Lemma 4, the operator H in Definition 4 is determined by the coordinate system. We call this the principal part of T in the coordinate system x. Let $x(x^*)$ be a C_n change of coordinates, $x(x^*) = x$, $x(y^*) = y$. By Lemmas 5 and 6, writing v for $\det(a_{ij})$,

$$\begin{aligned}
Hf(x) &= a(x)f(x) + \lim_{\epsilon\to 0}\int_{|x-y|>\epsilon} h(x, x-y)f(y)\,dy \\
&= a(x)f(x) - D(x)f(x) + \lim_{\epsilon\to 0}\int_{|x^*-y^*|>\epsilon} h(x, x-y)f(y)\,dy \\
&= a(x)f(x) - D(x)f(x) + \lim_{\epsilon\to 0}\int_{|x^*-y^*|>\epsilon} v(x^*)h(x, A(x^*-y^*))f(y)dy^* \\
&\quad + \int h(x, A(x^*-y^*))\{v(y^*) - v(x^*)\}f(y)\,dy^* \\
&\quad + \int R(x^*, y^*)f(y)v(y^*)dy^*.
\end{aligned}$$

Since $\phi(x^*)h(x(x^*), A(x^*-y^*))\{v(y^*) - v(x^*)\}\psi(y^*) = O(|x^*-y^*|^{1-k})$, it defines a completely continuous operator. Thus the principal part of T in the coordinate system x^* is the operator

$$\begin{aligned}
H'f(x^*) &= a(x(x^*))f(x^*) - D(x(x^*))f(x^*) \\
&\quad + \lim_{\epsilon\to 0}\int_{|x^*-y^*|>\epsilon} v(x^*)h(x(x^*), A(x^*-y^*))f(y^*)dy^*.
\end{aligned}$$

If we use this to compute the symbol, letting $\sum \xi_i dx_i = \sum \xi_i^* dx_i^*$, $\eta = A(\eta^*)$, we find

$$\begin{aligned}
\sigma(T)(p, \textstyle\sum \xi_i^* dx_i^*) &= a(x(p)) - D(x(p)) \\
&\quad + \lim_{\epsilon\to 0}\int_{|\eta^*|>\epsilon} \exp\{i\textstyle\sum \xi_j^*\eta_j^*\}h(x(p), A\eta^*)v(x^*)\,d\eta^* \\
&= a - D + \lim_{\epsilon\to 0}\int_{|\eta^*|>\epsilon} \exp\{i\textstyle\sum \xi_j\eta_j\}h(x(p), \eta)\,d\eta \\
&= a - D + \lim_{\epsilon\to 0}\int_{|\eta|>\epsilon} \exp\{i\textstyle\sum \xi_j\eta_j\}h(x(p), \eta)\,d\eta + D.
\end{aligned}$$

This proves Theorem 1.

THEOREM 2. *The correspondence of singular integral operators of type C_β^∞ with their symbols is a vector space homomorphism of the set of these operators onto the set of C_β^∞ functions on $CS(M)$.*

Proof. The only statement that needs to be proved is the fact that the homomorphism is onto. Let $F(p, \xi)$ be a C_β^∞ function on $CS(M)$. Let $\{\phi_i\}$

be a resolution of the identity as in section II-A above: $\sum \phi_i(p) = 1$, $0 \leqq \phi_i$, ϕ_i is in C_n and has support in a single coordinate neighborhood U_i. Let $\psi_i = 1$ on the support of ϕ_i, and have support in U_i, and let ψ_i be in C_n. Let x be a coordinate system in U_i. Then in terms of the basis $\{dx_i\}$ we can write, with $\xi = \sum \xi_i dx_i$ on the left, $\xi = (\xi_1, \cdots, \xi_k)$ on the right,

$$F(p, \xi) = a(p) + \sum b_{nm}(p) Y_{nm}(\xi).$$

Let $h(x, z) = \sum b_{nm}\gamma_n^{-1} Y_{nm}(z) \mid z \mid^{-k}$. Then set

$$T_i f(x) = \phi_i(x)\{a(x)f(x) + \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} h(x, x-y)\psi_i(y)f(y)\,dy\}.$$

The symbol of T_i is then $\phi_i F$, and the symbol of $\sum T_i$ is F. This proves Theorem 2.

Note that if L^p is given the norm $\| f \| = (\sum \| \phi_1^{1/p} f \|^p)^{1/p}$, then $\| \sum T_i f \| \leqq AM \| f \|$, where A is a constant depending on p, $\{\phi_i\}$, and the coordinate systems chosen, and M is an upper bound for F and its derivatives of order $2k$ with respect to cosphere variables. (See [2], Theorem 3).

C. Functional calculus of symbols.

THEOREM 3. *Let* $\beta \neq 1$. *Then* i) *if* T *is a* C_β^∞ *operator, so is* T^*, *and* $\sigma(T^*) = \sigma(T)^*$; ii) *if* $\sigma(T_1)\sigma(T_2) = \sigma(T)$, *then* $T - T_1T_2$ *is completely continuous and smoothing of order* $[\beta]$.

Proof. Let v be the volume element used to define the correspondence between L^q and the dual of L^p. Then $(\phi T \psi)^* = \psi^* H^* \phi^* + R^*$ (see Definition 4), and for f in L^q,

$$H^* f(x) = a^*(x)f(x) + \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} v(y)v(x)^{-1} h^*(y, y-x)f(y)\,dy.$$

Now if $h(x, z) = \sum a_{nm}(x) Y_{nm}(z) \mid z \mid^{-k}$, then

$$\sigma(T)(p, \sum \xi_i dx_i) = a(x(p)) + \sum a_{nm}(x(p))\gamma_n Y_{nm}(\xi)$$

and

$$\sigma(T)^* = a^* + \sum a_{nm}{}^* \gamma_n{}^* Y_{nm}(\xi) = a^* + \sum a_{nm}{}^* \gamma_n Y_{nm}(-\xi)$$

(for Y_{nm} is odd when n is odd, and even when n is even). The singular kernel associated with this symbol and this coordinate system is

$$H^\# f(x) = a^*(x)f(x) + \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} h^*(x, y-x)f(y)\ dy.$$

Thus it must be shown that

$$\lim_{\epsilon \to 0} \phi(x)v(x)^{-1} \int_{|x-y|>\epsilon} v(y)h^*(y, y-x)\psi(y)f(y)\,dy$$

differs by a completely continuous operator, smoothing of order $[\beta]$, from $\phi(x)\lim_{\epsilon\to 0}\int_{|x-y|>\epsilon} h^*(x,y-x)f(y)\psi(y)\,dy$. This is done by reverting again to spherical harmonics and the operators T_{nm}, as above. That $\phi\{cT_{nm}-T_{nm}c\}$ is completely continuous is proved above. That it is smoothing for $\beta>1$ is proved in [2], Theorem 5; Remarks (a) and (b) show it to be smoothing of order $[\beta]$. Assertion (i) is proved by noting that

$$\phi H^*\psi-\phi H^\#\psi=\sum\phi(T_{nm}a_{nm}^*-a_{nm}^*T_{nm})\psi.$$

To prove the second assertion, let $\{\phi_i\}$ be a resolution of the identity, with the support of ϕ_i in a coordinate neighborhood U_i with coordinates x^i. Let θ_i have support in U_i, and $\theta_i\equiv 1$ on an open set containing the support of ϕ_i; let ψ_i have support in U_i and $\psi_i\equiv 1$ on an open set containing the support of θ_i. Then

$$T-T_1T_2=\sum_i\psi_i(T-T_1T_2)\phi_i+\sum(1-\psi_i)(T-T_1T_2)\phi_i.$$

Now by Definition 4, $(1-\psi_i)\,T\phi_i$ is completely continuous and smoothing of order $[\beta]$. Also

$$(1-\psi_i)T_1T_2\phi_i=(1-\psi_i)T_1\theta_iT_2\phi_i+(1-\psi_i)T_1(1-\theta_i)T_2\phi_i.$$

Here $(1-\psi_i)T_1\theta_i$ and $(1-\theta_i)T_2\phi_i$ are completely continuous and smoothing of order $[\beta]$ and $T_2\phi_i$ and $(1-\psi_i)T_1$ preserve L_m^p, $m\leqq[\beta]$; thus $(1-\psi_i)T_1T_2\phi_i$ is completely continuous and smoothing of order $[\beta]$. Now $\psi_i(T-T_1T_2)\phi_i=(\psi_iT\phi_i-\psi_iT_1\theta_iT_2\phi_i)+\psi_iT_1(1-\theta_i)T_2\phi_i$. Here the second term is handled as above, and the first is (modulo completely continuous operators smoothing of order $[\beta]$) $\psi_iH\phi_i-\psi_iH_1\theta_iH_2\phi_i$, where H, H_1 and H_2 are Euclidean C_β^∞ operators whose symbols, on the image of U_i, are respectively $\sigma(T)$, $\sigma(T_1)$ and $\sigma(T_2)$. They can be extended so that $\sigma(H)=\sigma(H_1)\sigma(H_2)$ and for some R, $\sigma(H_2)(x,z)$ vanishes when $|x|\geqq R$. But $\psi_i(H-H_1\theta_iH_2)\phi_i\equiv\psi_i(H-H_1H_2)\phi_i+\psi_iH_1(1-\theta_i)H_2\phi_i$. For the first term we refer again to Theorem 5 of [2], and the complete continuity of $a(T_{nm}b-bT_{nm})$. In the second term $(1-\theta_i)H_2\phi_i$ is an operator with a bounded kernel in C_β, with bounded support in $E_k\times E_k$. Such an operator is completely continuous and smoothing of order $[\beta]$. This proves Theorem 3.

This shows that, if $\beta\neq 1$, the C_β^∞ operators form an algebra, and that σ is a homomorphism of this algebra onto the C_β^∞ functions on $CS(M)$.

THEOREM 4. (Compare [2], Theorem 6). *Let $k>2$ and $\beta\neq 1$. Let M be simply connected; or let $CS(M)$ have a cross-section. Then if $\sigma(T)$ is never zero, there is a C_β^∞ operator T' which is invertible, and $\sigma(T)=\sigma(T')$.*

Proof. Suppose M is simply connected and $k > 2$. Then $CS(M)$ is simply connected, and if $\sigma(T)$ is never zero, we can define $\sigma(T)^{1/n}$ so that $|1 - \sigma(T)^{1/n}| < \epsilon$ and the derivatives of order $2k$ of $\sigma(T)^{1/n}$ with respect to cosphere variables are also bounded by ϵ. Then by choosing n large enough, we can, by the comment after Theorem 2, find a C_β^∞ operator S with symbol $\sigma(T)^{1/n}$ satisfying $\| I - S \| < \frac{1}{2}$. Then S^n is invertible and $\sigma(S^n) = \sigma(T)$.

If $CS(M)$ has a cross-section $\phi\colon M \to CS(M)$, then

$$\{\sigma(T)(p, \phi(p))\}^{-1}\sigma(T)(p, \xi)$$

has an n-th root $G(p, \xi)$ approximating unity as above, so that G is the symbol of an invertible C_β^∞ operator S. Then $T' = \sigma(T)(p, \phi(p))S^n$ is invertible and $\sigma(T') = \sigma(T)$. This proves Theorem 4.

COROLLARY. *If M and T satisfy the hypotheses of Theorem 4, then the equation $Tf = g$ is equivalent to an equation of the form $f + Kf = g'$, where K is completely continuous.*

D. An example. Let M be a C_∞ Riemannian manifold with metric $r(p, q)$, p and q in M; and let V be a C_β vector field on M. Let $\phi(t)$ be a real function which is identically 1 in a neighborhood of $t = 0$, and vanishes outside a larger neighborhood, so that $\phi(r)r^2$ is a C_∞ function on $M \times M$. Let v be the volume element on M associated with the metric r, and let $Kf(p) = H(1, k)^{-1} \int_M \phi(r) r(p, q)^{1-k} f(q)\, dv_q$. Then VK is a C_β^∞ singular integral operator with symbol $\sigma(VK)(p, \xi) = i\{V(p), \xi/|\xi|\}$, where V is considered as a mapping from M into the tangent bundle, $|\xi|$ is the Riemannian length of the vector ξ in the cotangent bundle, and $\{\ ,\ \}$ indicates the bilinear form relating the tangent and the cotangent bundles. To show this, we need some facts about the metric r.

LEMMA 7. *If $r(x, y)$ is a C_∞ metric on E_k ($r^2(x, y)$ is in C_∞ on E_{2k}), then, for every n and m,*

$$r^m(x, y) = \sum_{i=0}^{n-1} a_i(x, x - y) + R(x, y),$$

where $a_0(x, z) = a_0(x, -z)$, $a_i(x, z)$ is in C_∞^∞ and homogeneous of degree $m + i$; and any derivative of $R(x, y)$ of order $j \leqq n$ is $O(|x - y|^{m+n-j})$.

COROLLARY. *$(\partial/\partial x + \partial/\partial y)^\alpha r(x, y) = O(|x - y|)$ for every α.*

Proof of corollary. In Lemma 7 take $m = 1$ and $n = |\alpha|$. Then

$r(x,y) = \sum a_i(x, x-y) + R(x,y)$. If $a_{i\alpha}(x,z) = (\partial/\partial x)^\alpha a_i(x,z)$, then $(\partial/\partial x + \partial/\partial y)^\alpha r(x,y) = \sum_i a_{i\alpha}(x,z) + (\partial/\partial x + \partial/\partial y)^\alpha R(x,y) = O(|x-y|)$.

Proof of the lemma. Let $r^2{}_\alpha(x,y) = (\partial/\partial y)^\alpha r^2(x,y)$. Then

$$r^2(x,y) = \sum_{i=2}^{n+1} \sum_{|\alpha|=i} r^2{}_\alpha(x,x)\,(y-x)^\alpha/\alpha! + P(x,y)$$
$$= \sum_{i=2}^{n+1} b_i(x, x-y) + P(x,y),$$

where $b_2(x,z) = b_2(x,-z)$, $b_i(x,z)$ is in $C_\infty{}^\infty$ and homogeneous of degree i, and derivatives of P of order $j \leqq n+1$ are $O(|x-y|^{n+2-j})$. Then

$$r^m(x,y) = b_2{}^{m/2}(x, x-y)\{1 + \sum_{i=3}^{n+1} b_i b_2{}^{-1} + P b_2{}^{-1}\}^{m/2}.$$

Using the Taylor expansion for $(1+x)^{m/2}$ with n terms and remainder, we get $r^m(x,y) = \sum a_i(x, x-y) + R(x,y)$, as required in the lemma.

Now let $r^{1-k}(x,y) = \sum_{j=0}^{n-1} a_j(x, x-y) + R(x,y)$, where $n-1 = [\beta]$. Then the arguments of Lemmas 2 and 3 and their corollary show that

$$\psi(x)\,(\partial/\partial x_i) \int \phi(r) a_0(x, x-y) f(y) v(y)\, dy = \psi(x) \int (\partial \phi a_0/\partial x_i) f v\, dy;$$

the multiplication factor $\lim_{\epsilon \to 0} \int_{|x-y|=\epsilon} \phi(r) a_0(x, x-y) \gamma_i \epsilon^{k-1} d\sigma_y$ is absent since a_0 is even on $|x-y| = \epsilon$ and γ_i is odd. In the same way,

$$\| \psi(\partial/\partial x_i) \int \phi(r) a_j(x, x-y) f(y) v(y)\, dy \|_{m+1} \leqq C \| f \|_m$$

for $m < [\beta]$ and $j \geqq 1$. Thus $\psi(\partial/\partial x_i) \int \phi r^{1-k} f v\, dy$ is a $C_\beta{}^\infty$ operator if ψ is in C_β.

It remains to calculate the symbol. Let the coordinates be geodesic at p, so that $a_0(x(p), z) = |z|^{1-k}$, and the kernel of VK is

$$H^{-1}\psi(p)\partial\, |x-y|^{1-k}/\partial x_i = H(1-k)^{-1}\psi(p)\,(x_i - y_i)/|x-y|^{k+1}.$$

But it has been shown that the transform of this operator kernel is $(-1)^{\frac{1}{2}}\xi_i/|\xi|$, (see IB), so that the symbol of $\psi(\partial/\partial x_i)K$ is

$$(-1)^{\frac{1}{2}}\psi\xi_i/|\xi| = (-1)^{\frac{1}{2}}\psi\{\partial/\partial x_i, \xi/|\xi|\},$$

using again the fact that the coordinates are normal at p.

A similar argument establishes $V_1V_2\int \phi(r)r^{2-k}f\,dv$ as a singular integral operator.

III. Differential Operators and Singular Integral Operators.

Our first goal is to develop a representation of an arbitrary differential operator on $L^p(M)$ in the form of a singular integral operator times a homogeneous differential operator related to the Laplace operator. We begin by discussing the latter operator, using a parametrix as tool. M will be a compact orientable manifold of class C_n $(n \geqq 4)$ and dimension k $(k \geqq 2)$ and with a Riemannian metric $r(p,q)$; $r^2(p,q)$ is a C_{n-1} function for small r.

A. The Laplace operator. Let Δ denote the Laplacian on M. For f in C_2

$$\Delta f = (1/v)\sum(\partial/\partial y_i)vg^{ij}\partial f/\partial y_j,$$

where $vdy_1\cdots dy_k$ is the volume element in terms of local coordinates, and g_{ij} are components of the metric tensor. Δ is a symmetric operator on its domain C_2, considered as a subspace of $L^2(M,v)$. Let $\bar{\Delta}$ be the closure of Δ as an operator on $L^2(M)$, obtained by closing the graph of Δ in $L^2(M)\times L^2(M)$. f in L^2 is in $D(\bar{\Delta})$, the domain of $\bar{\Delta}$, if there is a sequence $f_n \to f$ such that $f_n \times \Delta f_n \to f \times g$, where $f \times g$ represents an element of $L^2 \times L^2(M)$. It is easy to see that $L_2^2(M)$ is in $D(\bar{\Delta})$; for let f be in L_2^2 and have support in a single coordinate neighborhood (a harmless restriction). Then we can find a sequence of functions f_n in C_2 such that f_n and its derivatives approximate f in $L_2^2(E_k)$, hence in $L_2^2(M)$. (See, for instance, [2], Lemma 1.) Then both f_n and Δf_n converge in $L^2(M)$.

$\bar{\Delta}$ is symmetric, for $(\bar{\Delta}f, g) = \lim(\Delta f_n, g_n) = \lim(f_n, \Delta g_n) = (f, \bar{\Delta}g)$, where f_n and g_n are in C_2. If d denotes the gradient operator and f is in C_2, then $(\Delta f, f) = -(df, df)$, so for f in C_2 and $\lambda \geqq 0$, $((\lambda - \Delta)f, f) = (df, df) + \lambda(f,f) \geqq \lambda(f,f)$. In particular, all eigenvalues of $\bar{\Delta}$ are non-positive real numbers; for if $\bar{\Delta}\phi = \lambda\phi$, then $\lambda(\phi,\phi) = \lambda(\phi,\phi) + ((\bar{\Delta} - \lambda)\phi, \phi) \leqq 0$. We will show that $\bar{\Delta}$ is self-adjoint, has domain $L_2^2(M)$, and has pure point spectrum. The devices used will then be applied to the construction of Λ on the manifold.

Let $\phi(r)$ be identically 1 in $r < \delta$, and vanish for $r > R$. Pick R so that for each p in M the set $\{q: r(p,q) < 2R\}$ is contained in a single C_n coordinate system. Let $\epsilon_\lambda(p,q) = \phi(r(p,q))E_\lambda(r(p,q))$.

LEMMA 8. *If λ is not a negative real number and f is in $L^2(M)$, then $\epsilon_\lambda f(p) = \int_M \epsilon_\lambda(p,q) f(q)\, dv_q$ is in $L_2{}^2(M)$.*

Proof. Let f and ψ have support in a single coordinate neighborhood. By Lemma 2 and the estimates (2),

$$(\partial/\partial x_i) \int \psi(x)\phi(r) E_\lambda(r) f(y) v(y)\, dy = \int (\partial/\partial x_i)\{\psi(x)\phi(r)E_\lambda(r)\} f(y) v(y)\, dy;$$

and

$$(\partial^2/\partial x_i \partial x_j) \int \psi\phi E_\lambda f\, dv = \lim_{\epsilon\to 0} \int_{|x-y|>\epsilon} (\partial^2/\partial x_i \partial x_j)\{\psi(x)\phi(r)E_\lambda(r)\} f(y) v(y)\, dy - \psi(x) f(x) v(x) \lim_{\epsilon\to 0} \int_{|x-y|=\epsilon} (\partial E_\lambda/\partial x_i)\gamma_j \epsilon^{k-1}\, d\sigma_y,$$

if these limits exist as required by Lemma 2. But by (2) and Lemma 7,

$$\partial E_\lambda/\partial x_i = E_\lambda'(r)\partial r/\partial x_i = b_1(x, x-y) + O(|x-y|^{2-k}),$$

and $\partial^2 E_\lambda/\partial x_i \partial x_j = b_0(x, x-y) + O(|x-y|^{1-k})$, where $b_j(x,z)$ $(j=0,1)$ is homogeneous of degree $j-k$ in z, which proves Lemma 8.

LEMMA 9. *If f is in L^2, then*

$$(\bar{\Delta} - \lambda)\epsilon_\lambda f(p) = f(p) + \int K_\lambda(p,q) f(q)\, dv_q,$$

where $K_\lambda(p,q) = (\Delta - \lambda)_p \epsilon_\lambda(p,q)$ for $p \neq q$. $K_\lambda(p,q)$ is $O(|x-y|^{2-k})$.

Proof. We show first the order of $K_\lambda(p,q)$ by reference to coordinates geodesic at q. In such a system $x(q)=0$, $g_{ij}(0)=\delta_{ij}$, $(\partial g_{ij}/\partial y_k)(0)=0$, $(\partial v/\partial y_k)(0)=0$, and $r^2(p,q) = \sum x_i^2(p)$. Thus

$$\Delta h(x) = h(x)\cdot O(1) + \sum \partial h/\partial x_i \cdot O(|x|) + \sum_{i\neq j} \partial^2 h/\partial x_i \partial x_j \cdot O(|x|^2) + \sum \partial^2 h/\partial x_i{}^2.$$

Using this with the estimates (2), we see $(\Delta - \lambda)_p \epsilon_\lambda(p,q) = O(|x|^{2-k})$.

Now from the form of the derivatives of $\epsilon_\lambda f(p)$ obtained in the proof of Lemma 8 we get

$$(\bar{\Delta}-\lambda)\epsilon_\lambda f = \lim_{\epsilon\to 0}\int_{|x-y|>\epsilon}\{(\Delta-\lambda)\epsilon_\lambda\}f\,dv$$
$$-f(x)v(x)\sum g^{ij}(x)\lim_{\epsilon\to 0}\int_{|x-y|=\epsilon}(\partial E_\lambda/\partial x_i)\gamma_j\epsilon^{k-1}\,d\sigma_y.$$

By what we have just shown, the $\lim_{\epsilon\to 0}$ on the first integral on the right is superfluous. To evaluate the function which multiplies f, let the coordinates be geodesic at x, so that $v(x)=1$, $g^{ij}(x)=\delta_{ij}$, and

$$\partial E_\lambda(r(x,y))/\partial x_i = \omega_k^{-1}r^{1-k}\partial r/\partial x_i = -\omega_k^{-1}\,|\,x-y\,|^{1-k}\gamma_i.$$

Thus

$$v\sum g^{ij}\lim_{\epsilon\to 0}\int_{|x-y|=\epsilon}(\partial E_\lambda/\partial x_i)\gamma_j\epsilon^{k-1}\,d\sigma_y = -\omega_k^{-1}\int_{|x-y|=\epsilon}\sum\gamma_i^2\,d\sigma_y = -1,$$

which proves the lemma.

Now we show that for $\mathrm{Re}(\lambda)\geqq 0$ and λ sufficiently large, $\bar{\Delta}-\lambda$ has a kernel operator as its inverse. Choose N so that

$$K_\lambda^{(N)}(p,q) = \int\cdots\int K_\lambda(p,p_1)K_\lambda(p_1,p_2)\cdots K_\lambda(p_{N-1},q)\,dv_{p_1}\cdots dv_{p_{N-1}}$$

is bounded. Set

$$F_\lambda(p,q) = \sum_0^{N-1}(-1)^j\int\epsilon_\lambda(p,p_1)K_\lambda^{(j)}(p_1,q)\,dv_{p_1}$$
$$+\int\epsilon_\lambda(p,p_1)f(p_1,q)\,dv_{p_1}, \tag{5}$$

where $K_\lambda^{(j)}$ is the j-fold iterate of K_λ. $F_\lambda(p,q)$ is a kernel defining a bounded operator on $L^2(M)$ if $\|f(\cdot,q)\|_2$ is uniformly bounded. In the sense of operators,

$$(\bar{\Delta}-\lambda)F_\lambda = I+(-1)^{N-1}K_\lambda^{(N)}+f+K_\lambda f,$$

where I is the identity, and f the operator with kernel $f(p,q)$. Hence in order for F_λ to be the Green's kernel for $\bar{\Delta}-\lambda$, it is necessary and sufficient that f satisfy

$$(-1)^{N-1}K_\lambda^{(N)}+f+K_\lambda f = 0. \tag{6}$$

Now from the estimates (3) and the form of $(\Delta-\lambda)\epsilon_\lambda$ deduced in Lemma 9, a simple integration shows that $|||\,\epsilon_\lambda\,||| \leqq c/|\,\lambda\,|$, and $|||\,K_\lambda\,||| \leqq c/|\,\lambda\,|$, where $|||\,F\,|||$ indicates the sup over $1\leqq r\leqq\infty$ of the norm of the operator with kernel $F(p,q)$ on $L^r(M)$. For on each coordinate neighborhood these operators

are dominated by convolution with a function whose norm in $L^1(E_k)$ is bounded by $c/|\lambda|$. Hence, for $\lambda \geqq L$, $|||K_\lambda||| \leqq \frac{1}{2}$ and

$$F_\lambda = \epsilon_\lambda(1 + \sum_{m=1}^{\infty}(-1)^m K_\lambda^{(m)}), \tag{7}$$

where the series of operators converges in norm, and the series of kernels converges uniformly in the manifold variables and in $\lambda > L$.

PROPOSITION. *The closure of the Laplacian on M is a self-adjoint operator with domain $L_2^2(M)$. It has pure point spectrum, lying on $\lambda \leqq 0$, with a limit only at ∞. The eigenfunction span $L^2(M)$, and the eigenspace of $\lambda = 0$ consists of the constant functions.*

Proof. For f in L^2, $(\bar{\Delta} - L)F_L f = f$; since $\bar{\Delta}$ is symmetric, $F_L^*(\bar{\Delta} - L)g = g$ for g in L_2^2. But for f in L^2, $F_L f$ is in L_2^2, and $F_L f = F_L^*(\bar{\Delta} - L)F_L f = F_L^* f$. Thus F_L is a Hermitian right and left inverse for $\bar{\Delta} - L$; since it is completely continuous, the eigenvalues for $\bar{\Delta} - L$ have a limit only at ∞, and the eigenfunctions span $L^2(M)$. If $\bar{\Delta}\phi = 0$, $0 = (\bar{\Delta}\phi, \phi) = -(d\phi, d\phi)$, so that ϕ has zero (weak) derivatives, from which it can be shown that ϕ is constant. (We do not prove this, nor do we use the fact.) The statement about the domains follows from the fact that range of $F_L \subset L_2^2(M) \subset$ domain of $\bar{\Delta} - L \subset$ range of F_L. The first inclusion is from Lemma 8, the second from a previous remark, and the last from the fact that F_L is the inverse of $\bar{\Delta} - L$.

By assuming one more order of differentiability for M, we could show by this approach that the eigenfunctions of $\bar{\Delta}$ are all in C_2; for by iterating F_L, any function in L^2 can be mapped into C_2.

B. A semi-group of operators associated with $L - \Delta$. From now on we will assume M to be a compact orientable C_∞ Riemannian manifold; everything could be done by assuming less, but the frequent statement of differentiability assumptions would be distracting. We denote by Δ the closed self-adjoint operator discussed in the previous section. ϵ_λ, F_λ and K_λ are as above, but L is chosen so that $|||\epsilon_\lambda||| < \frac{1}{2}$ and $|||K_\lambda||| < \frac{1}{2}$ for $|\lambda| \geqq L/2$.

As in the Euclidean case, we can define operators J^α on M by a contour integral:

$$\begin{aligned} J^\alpha = (L - \Delta)^{-\alpha/2} &= \tfrac{1}{2\pi i}\int_C \lambda^{-\alpha/2}(\lambda - L + \Delta)^{-1} d\lambda \\ &= \tfrac{1}{2\pi i}\int_C \lambda^{-\alpha/2} F_{L-\lambda} d\lambda, \end{aligned} \tag{8}$$

where C is the path $\mathrm{Re}(\lambda) = L/2$ traversed from top to bottom, and $\lambda^{-\alpha/2}$ is analytic except on $\lambda \leqq 0$. The integrand is a norm-continuous function of λ:

$$(9) \qquad F_{\lambda+h} - F_\lambda = F_\lambda\{(-\lambda + \Delta - h + h)F_{\lambda+h} - I\} = F_\lambda h F_{\lambda+h}.$$

This with the fact that $|||F_\lambda||| = O(1/\lambda)$ shows that the integral converges in operator norm on every $L^p(M)$, $1 \leqq p \leqq \infty$, for $\mathrm{Re}(\lambda) > 0$. Since the Fourier transform is not available to show the properties of this operator, we resort to a Banach algebra approach.

Since $F_\lambda^* = F_{\lambda^*}$ and $F_{\lambda_1}F_{\lambda_2} = F_{\lambda_2}F_{\lambda_1}$, the algebra A of operators on $L^2(M)$ generated by the F_λ ($\mathrm{Re}(\lambda) \geqq L/2$) and I is commutative and self-adjoint, and isometric to its representing function algebra. Let M be a maximal ideal of A, and $\hat{H}(M) = M(H)$ be the representing function of the element H of A. We show that

$$(10) \qquad \hat{J}^\alpha(M) = (-\hat{F}_L(M))^{\alpha/2}.$$

From (9), $\hat{F}_\lambda(M) = 0$ if $\hat{F}_L(M) = 0$, so for these M, (10) holds. Consider $\hat{F}_L(M) \neq 0$. For $-\lambda^{-1}$ not in the spectrum of F_L, $\hat{F}_{L-\lambda}(M) = \hat{F}_L(M)\{1 + \lambda\hat{F}_L(M)\}^{-1}$, by (9). Now

$$\hat{J}^\alpha(M) = \tfrac{1}{2\pi i}\int_C \lambda^{-\alpha/2}\hat{F}_L(M)\{1 + \lambda\hat{F}_L(M)\}^{-1}\,d\lambda.$$

If we set $\mu = 1/\lambda$, then μ traverses a circle C' counter-clockwise as λ traverses C, and

$$\hat{J}^\alpha(M) = (-\hat{F}_L(M)/2\pi i)\int_{C'} \mu^{(\alpha/2)-1}\{\mu + \hat{F}_L(M)\}^{-1}\,d\mu = \{-\hat{F}_L(M)\}^{\alpha/2},$$

since $-\hat{F}_L(M)$ is inside C'. Thus the J^α form a semi-group of bounded operators on $L^2(M)$ with $J^2 = (L - \Delta)^{-1}$.

C. Further analysis of $J = J^1$, and representation of vector fields. In this section it is shown that $J = J^1$ is a completely continuous operator and smoothing of all orders; and that for any C_β vector field V, VJ is a C_β^∞ singular integral operator. Finally, the representation of $V_1 \cdots V_n$ as $H(L - \Delta)^{n/2}$, H a singular integral operator, is obtained.

The first step is to change the contour of integration for J to a curve enclosing the negative real axis; this can be justified by considering the representing algebra of A, the separable Banach algebra with identity generated by the F_λ for $\mathrm{Re}(\lambda) \geqq L/2$. Then a change of variables allows us to write

$$J = (\tfrac{1}{\pi}) \int_0^\infty s^{-\frac{1}{2}} F_{L+s}\, ds.$$

That J is completely continuous follows from the fact that this integral can be considered as a strong vector-valued integral in the Banach space generated by the F_λ for $\lambda = L/2$, hence lies in that space of completely continuous operators.

THEOREM 5. *If V is a C_β vector field then VJ is a C_β^∞ singular integral operator with symbol*

$$\sigma(VJ)(p,\xi) = (-i)\{V, \xi/|\xi|\}.$$

The theorem depends on some facts about the operators ϵ_λ and K_λ. In what follows, $\| \ \|_{m,n}$ indicates the norm of an operator from L_m^p to L_n^p.

i) $\| V\epsilon_\lambda \|_{m,m+1} < A(m)$ $(m \geqq 0)$. A does not depend on λ.

ii) $\| K_\lambda \|_{m,m+1} < B(m)$ $(m \geqq 0)$, B independent of λ.

iii) $\| K_\lambda \|_{m,m} < B(m)/\lambda$ $(m \geqq 0)$, B independent of λ.

Proof of (i). Let $X(t) = 1$ for $t < 1, 0$ for $t > 1$; and $r^{2-k}(x,y) = b(x, x-y) + R(x,y)$, where $b(x,z)$ is the leading term of the expansion in Lemma 7; and let $\partial^2 b(x,z)/\partial x_i \partial x_j = b_{ij}(x,z)$. Then by Lemma 8,

$$(\partial/\partial x_j) \int \psi(\partial\epsilon_\lambda/\partial x_i) f\, dv = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} (\partial/\partial x_j)(\psi \partial\epsilon_\lambda/\partial x_i) f(y) v(y)\, dy - m(x) f(x),$$

where m is independent of λ. Let $\mu = \lambda^{\frac{1}{2}}$ and dissect the above expression as follows:

$$\lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} X(\mu\,|\,x-y\,|) b_{ij}(x, x-y) f(y) v(y)\, dy + \int X(\mu\,|\,x-y\,|)\{b_{ij}(x,z) - (\partial/\partial x_j)(\psi\partial\epsilon_\lambda/\partial x_i)\} f v\, dy + \int \{1 - X(\mu\,|\,x-y\,|)\}(\partial/\partial x_j)(\psi\partial\epsilon_\lambda/\partial x_i) f v\, dy - m(x) f(x).$$

That the first operator, acting on L^p, has a norm with a bound independent of μ is proved in [1]; the norm of the second is estimated by (2), independently of μ; and the third is treated similarly by (3).

Thus $\| V\epsilon_\lambda \|_{0,1} < C$, where C is independent of λ. To complete the proof of (i), we refer to remarks (b) and (c) and Lemma 2. Let C_α be as in

Remark (c) (*I-D*), $\alpha' = (\alpha_1, \cdots, \alpha_j + 1, \cdots, \alpha_k)$. Then repeated application of Lemma 2 shows that $C_\alpha((\partial/\partial x_j)\psi(\partial/\partial x_i)\epsilon_\lambda - \psi(\partial/\partial x_i)\epsilon_\lambda(\partial/\partial x_j))$ is the operator with kernel $(\partial/\partial x + \partial/\partial y)^{\alpha'}(\psi\partial\phi E_\lambda/\partial x_i)$. But by the Corollary of Lemma 7 and the estimates (3), $|(\partial/\partial x + \partial/\partial y)^{\alpha'}\psi\partial\phi E_\lambda/\partial x_i| < P_n(\mu r)e^{-\mu r}r^{1-k}$ or $< P_n(\mu r)e^{-\mu r}r^{1-k}(\mu r)^{\frac{1}{2}(k-3)}$, so that this operator has a norm which is $O(1/\mu)$. Thus Remarks (b) and (c) show that $\| V\epsilon_\lambda \|_{m,m+1} < A(m)$, with A independent of λ. For (ii), $\| K_\lambda \|_{0,1}$ is found, by direct application of Lemma 2 and the estimates (3), to be $O(\lambda^{-\frac{1}{2}})$, and this is extended to $\| K_\lambda \|_{m,m+1}$ exactly as in (i). (iii) is proved simply by noting that

$$|(\partial/\partial x + \partial/\partial y)^\alpha \phi K_\lambda| < P_n(\mu r)e^{-\mu r}r^{2-k}(\mu r)^{\frac{1}{2}(k-3)} \text{ or } < P_n(\mu r)e^{-\mu r}r^{2-k},$$

and applying Remark (c).

Now to prove the theorem, let $[\beta] = n$ and write

$$VJ = (\tfrac{1}{\pi})V\int_0^\infty s^{-\frac{1}{2}}\epsilon_s\,ds - (\tfrac{1}{\pi})V\int_0^L s^{-\frac{1}{2}}\epsilon_s\,ds - (\tfrac{1}{\pi})V\int_0^\infty \{s^{-\frac{1}{2}} - (L+s)^{-\frac{1}{2}}\}\epsilon_{L+s}\,ds$$

$$\text{(iv)} \qquad + (\tfrac{1}{\pi})\sum_1^n (-1)^m V\int_0^\infty \epsilon_{L+s}K_{L+s}{}^{(m)}s^{-\frac{1}{2}}\,ds$$

$$+ (\tfrac{1}{\pi})V\int_0^\infty \sum_1^\infty (-1)^{n+m}\epsilon_{L+s}K_{L+s}{}^{(n+m)}s^{-\frac{1}{2}}\,ds.$$

Now from (4) and the definition of ϵ_s,

$$(\tfrac{1}{\pi})\int_0^\infty s^{-\frac{1}{2}}\epsilon_s(r(x,y))\,ds = -H(1,k)^{-1}\phi(r)r^{1-k}.$$

Fubini's theorem shows that $\int_0^\infty s^{-\frac{1}{2}}\epsilon_s\,ds$, considered as an operator integral, is the operator with kernel $\int_0^\infty s^{-\frac{1}{2}}\epsilon_s(r(x,y))\,ds$; thus the first term in (iv) is $-VH(1,k)^{-1}\phi(r)r^{1-k}$, which is the singular integral operator of example II D. In the other terms the differentiation may be carried under the integral and summation since the differentiated integral converges in operator norm:

$$-(\partial g/\partial x_i, \int_0^\infty \sum_1^\infty s^{-\frac{1}{2}}\epsilon_{L+s}K_{L+s}{}^{(m)}\,ds)$$

$$= \lim\{-(\partial g/\partial x_i, \sum_1^N \sum_1^M s_n{}^{-\frac{1}{2}}\epsilon_{L+s_n}K_{L+s_n}{}^{(m)}\Delta s_n)\}$$

$$= \lim(g, \sum^N \sum^M s_n{}^{-\frac{1}{2}}(\partial/\partial x_i)\epsilon_{L+s_n}K_{L+s_n}{}^{(m)}\Delta s_n)$$

$$= (g, \int_0^\infty \sum_1^\infty s^{-\frac{1}{2}}(\partial/\partial x_i)\epsilon_{L+s}K_{L+s}{}^{(m)}\,ds).$$

Now the second and third terms are bounded from $L_m{}^p$ to $L_{m+1}{}^p$ by (i).

The fourth term is also bounded from L_m^p to L_{m+1}^p by (i) and (iii). In the last term,

$$\| V\epsilon_{L+s}K_{L+s}{}^{(n+m)} \|_{0,n} \leqq (\| V\epsilon_{L+s}K_{L+s}{}^{(n-1)} \|_{0,n})(\| K_{L+s} \|_{0,0})(\| K_{L+s} \|_{0,0})^m$$
$$\leqq A(n)(c/(L+s))(\tfrac{1}{2})^m,$$

by the choice of L, so that the integrated sum converges to an operator from L^p to L_n^p. This proves Theorem 5.

Since for each vector field V in C_∞, VJ is bounded from L_m^p to L_m^p, J is smoothing of all orders.

LEMMA 10. *If V is a C_β vector field, $VJ - JV$ is smoothing of order $[\beta]$, if $\beta \neq 1$.*

Proof. $J = -H(1,k)^{-1}\phi(r)r^{1-k} + R$. In Theorem 5 it was shown that VR was smoothing of order $[\beta]$. Similarly, RV is smoothing of order $[\beta]$. That the same is true of $VK - KV$, where K is the operator with kernel $\phi(r)r^{1-k}$, is checked with the help of Lemma 7, Lemma 3, and the Corollary of the latter.

Definition 6. For f in L_m^p, $m \geqq 1$, $\Lambda f = (L - \Delta)Jf$.

If f is in L_m^p, $m > 1$, $\Lambda f = J(L - \Delta)f$, since J and $L - \Delta$ commute. It is clear that Λ maps L_m^p into L_{m-1}^p, and that J is its two-sided inverse.

THEOREM 6. *If $\beta \neq 1$ and V_i is a $C_{\beta+i-1}$ vector field, then $V_1 \cdots V_m = H\Lambda^m$, where H is a $C_\beta{}^\infty$ singular integral operator with symbol*

$$(-(i)^m \prod_{k=1}^{m} \{V_k, \xi/|\xi|\}.$$

Proof. $V_1 \cdots V_n = V_1 \cdots V_n J^n \Lambda^n = (V_1 J) \cdots (V_n J)\Lambda^n$

$$+ \sum_{j=1}^{n-1} V_1 \cdots V_j (V_{j+1}J^j - J^j V_{j+1}) J V_{j+2} \cdots J V_n \Lambda^n$$

By Theorem 3 and Theorem 5 $(V_1 J) \cdots (V_n J)$ is a $C_\beta{}^\infty$ operator with the prescribed symbol. We show that

$$\sum V_1 \cdots V_j (V_{j+1}J^j - J^j V_{j+1})(JV_{j+2}) \cdots (JV_n)$$

maps L_m^p into L_{m+1}^p by showing that $V_{j+1}J^j - J^j V_{j+1}$ maps L_m^p into L_{m+j}^p. This is done by Lemma 8 and the formula $VJ^j - J^j V = \sum_{k=1}^{j} J^{k-1}(VJ - JV)J^{j-k}$. This establishes Theorem 6.

THEOREM 7. *The spaces L_m^p $(1 < p < \infty)$ are topologically isomorphic. Any operator on M which is smoothing is also completely continuous.*

Proof. The first statement is proved with the isomorphism J and its inverse Λ. To prove the second, suppose T maps L^p into L_1^p. Then ΛT is bounded, and $T = J\Lambda T$ is completely continuous.

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CHARACTERIZATIONS OF RIEMANN n-SPHERES.*

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Introduction. One of the most interesting classical results in differential geometry in the large is a theorem of Liebmann [14], which states that in ordinary Euclidean space E^3 a closed surface with constant Gaussian curvature is a sphere. For this theorem Hilbert ([8]; or [9], pp. 231-240) gave an ingenious proof, which has since superseded the original one given by Liebmann. It is also well known that in the space E^3 a closed convex surface with constant mean curvature is a sphere. These theorems were extended by Süss [17] to convex hypersurfaces in a Euclidean space E^n of dimension $n \geqq 3$. Recently, Hsiung [10, 11] and Aleksandrov [1, 2] [2] have further extended the results of Süss to hypersurfaces imbedded in a space E^n as well as in an n-dimensional Riemannian manifold of constant Riemannian curvature. In Hsiung's papers the hypersurfaces satisfy a condition weaker than convexity and are so called star-shaped, and in Aleksandrov's the elementary symmetric functions of the principal curvatures of the hypersurfaces are replaced by some more general functions. The purpose of this paper is to use a new method to complete the latter case of Hsiung's work and to further study a more general case in which hypersurfaces are imbedded in a general Riemannian manifold of dimension $\geqq 3$.

In § 1, we first define the vector product of $m-1$ tangent vectors of a Riemannian manifold V^m of dimension $m \geqq 3$ at a point P, and then introduce E. Cartan's system of exterior differential forms for the manifold V^m.

In § 2, from the system of exterior differential forms for the Riemannian manifold V^m introduced in § 1 we geometrically deduce a system for an n-dimensional submanifold V^n of the manifold V^m, where $m > n \geqq 2$. The first and second fundamental forms of the submanifold V^n are obtained, and the generalized covariant differentiation is introduced.

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[2] The authors are indebted to the referee for calling their attention to the second paper of Aleksandrov.

As a special case of § 1 and § 2, the next section is devoted to the establishment of the fundamental formulas and definitions for a hypersurface V^n imbedded in a Riemannian manifold V^{n+1} of dimension $n+1 \geqq 3$. In particular, the definition of Riemann n-sphere is given, and a differential form concerning a general mean curvature M_α $(\alpha = 1, \cdots, n)$ of the hypersurface V^n is derived by using the combined operator $\otimes$ of the vector product of tangent vectors of the manifold V^{n+1} and the exterior product of differentials (for this operator see, for instance, [12]).

§ 4 contains the statements of two lemmas, which are fundamentally important in the proofs of our main theorems.

Let V^{n+1} be a Riemannian manifold of dimension $n+1 \geqq 3$ and constant Riemannian curvature such that there is a normal coordinate system S of Riemann at a fixed point O covering the whole manifold V^{n+1}. In § 5, we first derive n integral formulas for an orientable hypersurface V^n of class C^3 with boundary V^{n-1} of dimension $n-1$ imbedded in the manifold V^{n+1}, and then for a closed hypersurface V^n to be a Riemann n-sphere we deduce some simple conditions on the mean curvatures M_α $(\alpha = 1, \cdots, n)$ of the hypersurface V^n.

The objective of § 6 is the same as that of § 5, except that in § 6 the Riemannian manifold V^{n+1} is a general one instead of one with constant Riemannian curvature. However, not all results in § 5 are special cases of those in § 6.

1. Riemannian manifolds. Throughout this paper the ranges of indices are given as follows unless stated otherwise:

$$\begin{aligned} &1 \leqq i, j, k, \cdots \leqq m, \\ &1 \leqq \alpha, \beta, \gamma, \cdots \leqq n, \\ &n+1 \leqq A, B, C, \cdots \leqq m \qquad (m > n). \end{aligned} \tag{1.1}$$

We shall also follow the usual tensor convention that when the same letter appears in any term as a subscript and superscript, it is understood that this letter is summed for all the possible values.

Let V^m be a Riemannian manifold of dimension $m \geqq 3$ and class C^3, y^i a set of local coordinates of a point P on the manifold V^m, and $a_{ij}dy^i dy^j$ the fundamental form of the manifold V^m, where $a_{ij} = a_{ji}$ and the matrix (a_{ij}) is positive definite so that the determinant $|a_{ij}| = a$ is positive.

Now let A_r $(r = 1, \cdots, m-1)$ be $m-1$ tangent vectors of the manifold V^m at the point P, and let $A_r{}^i$ $(r = 1, \cdots, m-1)$ be the contravariant com-

ponents of the vector A_r in the local coordinates $y^1, \cdots, y^m$ of the point P. Let $A_1 \times \cdots \times A_{m-1}$ denote the vector product of the $m-1$ vectors $A_1, \cdots, A_{m-1}$, which is defined to be the tangent of the manifold V^m at the point P whose j-th contravariant component is (see, for instance, Hsiung [11])

$$(1.2)\quad (A_1 \times \cdots \times A_{m-1})^j = (-1)^{m-1} a^{-\frac{1}{2}} \begin{vmatrix} \delta_1{}^j & \delta_2{}^j & \cdots & \delta_m{}^j \\ a_{i_1} A_1{}^i & a_{i_2} A_1{}^i & \cdots & a_{im} A_1{}^i \\ \cdots & \cdots & \cdots & \cdots \\ a_{i_1} A_{m-1}{}^i & a_{i_2} A_{m-1}{}^i & \cdots & a_{im} A_{m-1}{}^i \end{vmatrix},$$

where $\delta_i{}^j$ are the Kronecker deltas.[3] Let I be a tangent vector of the manifold V^m at the point P with contravariant components I^i in $y^1, \cdots, y^m$. From the definition of the scalar product of any two vectors, A_r and A_s, namely,

$$(1.3)\qquad A_r \cdot A_s = a_{ij} A_r{}^i A_s{}^j,$$

it follows that the scalar product of the two vectors I and $A_1 \times \cdots \times A_{m-1}$ is given by

$$(1.4)\qquad I \cdot A_1 \times \cdots \times A_{m-1} = (-1)^{m-1} a^{\frac{1}{2}} \, | I, A_1, \cdots, A_{m-1} |,$$

where $| I, A_1, \cdots, A_{m-1} |$ is a determinant, the elements of each of whose columns are the contravariant components of the vector indicated. Thus from equation (1.4) it follows immediately that the vector $A_1 \times \cdots \times A_{m-1}$ is orthogonal to each of the $m-1$ vectors $A_1, \cdots A_{m-1}$.

Now consider a frame $Pe_1 \cdots e_m$, where $e_1, \cdots, e_m$ form an ordered set of m mutually orthogonal unit tangent vectors of the manifold V^m at a point P so that

$$(1.5)\qquad e_i \cdot e_j = a_{hk} e_i{}^h e_j{}^k = \delta_{ij},$$

where δ_{ij} are the Kronecker deltas. The position vector Y of the point P is defined to be the tangent vector of the manifold V^m at the point P whose contravariant components are the local coordinates $y^1, \cdots, y^m$ of the point P. Then we can write

$$(1.6)\qquad dY = \omega^i e_i,$$

$$(1.7)\qquad de_i = \omega_i{}^j e_j,$$

where d denotes the exterior differentiation, and ω^i, $\omega_i{}^j$ are Pfaffian forms in the y's satisfying

$$(1.8)\qquad \omega_i{}^j + \omega_j{}^i = 0.$$

[3] The authors wish to thank Professor E. H. Cutler for some discussions about the form of the formula (1.2).

By means of exterior differentiation and equations (1.5), (1.7) and $d^2Y = d(dY) = 0$, we can obtain the following equations of structure of the manifold V^m (see Cartan [4], p. 222):

$$d\omega^i = \omega^j \wedge \omega_j{}^i, \tag{1.9}$$

$$d\omega_i{}^j = \omega_i{}^k \wedge \omega_k{}^j + \tfrac{1}{2} R^j{}_{ikl}\omega^k \wedge \omega^l, \tag{1.10}$$

where $\wedge$ denotes the exterior multiplication and $R^j{}_{ikl}$ are the components of the Riemann-Christoffel tensor of the manifold V^m.

2. Submanifolds of Riemannian manifolds. Let V^n be a Riemannian manifold of dimension n $(2 \leqq n < m)$ and class C^3 imbedded in a Riemannian manifold V^m defined in § 1. The position vector Y of a point P on the submanifold V^n can be given by the vector equation $Y = F(x^1, \cdots, x^n)$, where the n parameters $x^1, \cdots, x^n$ take values in a simply connected domain D of the n-dimensional real number space, the components of the vector $F(x^1, \cdots, x^n)$ are of class C^3, and the Jacobian matrix $(\partial Y/\partial x^\alpha)$ is of rank n at all points of the domain D. Let the first fundamental form of the submanifold V^n at a point P be

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta, \tag{2.1}$$

where the matrix $(g_{\alpha\beta})$ is positive definite, so that the determinant $|g_{\alpha\beta}| = g$ is positive. It is known that

$$g_{\alpha\beta} = a_{ij} y^i{}_{,\alpha}\, y^j{}_{,\beta}, \tag{2.2}$$

where $y^i{}_{,\alpha} = \partial y^i/\partial x^\alpha$.

To study the submanifold V^n we consider the submanifold of the manifold of frames $Pe_1 \cdots e_m$ as defined in § 1, such that $e_1, \cdots, e_n$ are tangent vectors of the submanifold V^n at the point P. Denoting by the same symbols the forms on this submanifold of frames induced by the identity mapping, we have, from equation (1.6),

$$\omega^A = 0, \tag{2.3}$$

and therefore, from equation (1.9),

$$d\omega^A = \omega^\alpha \wedge \omega_\alpha{}^A = 0. \tag{2.4}$$

By a lemma of E. Cartan ([3], p. 11) on exterior algebra, equation (2.4) implies that for each value of A

$$\omega_\alpha{}^A = b^A{}_{\alpha\beta}\omega^\beta, \tag{2.5}$$

where $b^A{}_{\alpha\beta} = b^A{}_{\beta\alpha}$. From equations (1.5), (1.6), (1.7), (1.8), (2.3) and (2.5), it follows that the first and second fundamental forms of the submanifold V^n at the point P are respectively given by

$$dY \cdot dY = \sum_{\alpha=1}^{n} (\omega^\alpha)^2, \tag{2.6}$$

$$\Phi_A = -dY \cdot de_A = b^A{}_{\alpha\beta}\omega^\alpha\omega^\beta. \tag{2.7}$$

Now we are in a position to introduce the generalized covariant differentiation, which is useful for studying submanifolds of Riemannian manifolds. Let $A^i{}_{j\alpha}$ be a mixed tensor of the second order in the y's and a covariant vector in the x's, as indicated by the Latin and Greek indices. Then following Tucker [19], the generalized covariant derivative of $A^i{}_{j\alpha}$ with respect to the x's is defined by

$$A^i{}_{j\alpha;\beta} = \partial A^i{}_{j\alpha}/\partial x^\beta + \{{}^{\ i}_{k\ l}\}A^k{}_{j\alpha}y^l{}_{,\beta} - \{{}^{\ k}_{j\ l}\}A^i{}_{k\alpha}y^l{}_{,\beta} - \{{}^{\ \gamma}_{\alpha\ \beta}\}A^i{}_{j\gamma}, \tag{2.8}$$

where the Christoffel symbols $\{{}^{\ i}_{j\ k}\}$ with Latin indices are formed with respect to the a_{ij} and the y's, and those $\{{}^{\ \alpha}_{\beta\ \gamma}\}$ with Greek indices with respect to the $g_{\alpha\beta}$ and the x's. It should be noted that this definition of generalized covariant differentiation can be applied to any tensor in the x's and y's, and that the generalized covariant differentiation of sums and products of tensors obeys the ordinary rules. If a tensor is one with respect to the x's only, so that only Greek indices appear, its generalized covariant derivative is the same as its covariant derivative with respect to the x's. Furthermore, in generalized covariant differentiation the fundamental tensors a_{ij} and $g_{\alpha\beta}$ can be treated as constants.

Since y^i is an invariant for transformation of the x's, its generalized covariant derivative is the same as its covariant derivative with respect to the x's, so that

$$y^i{}_{;\alpha} = y^i{}_{,\alpha} = \partial y^i/\partial x^\alpha. \tag{2.9}$$

By equation (2.8) the generalized covariant derivative of $y^i{}_{;\alpha}$ is

$$y^i{}_{;\alpha\beta} = \partial^2 y^i/\partial x^\beta\partial x^\alpha + \{{}^{\ i}_{j\ k}\}y^j{}_{,\alpha}y^k{}_{,\beta} - \{{}^{\ \gamma}_{\alpha\ \beta}\}y^i{}_{,\gamma}, \tag{2.10}$$

which is symmetric in the indices α and β.

3. Hypersurfaces of Riemannian manifolds. This section is devoted to the establishment of the fundamental formulas for a hypersurface V^n imbedded in a Riemannian manifold V^{n+1} of dimension $n+1 \geqq 3$. For this purpose we shall use the results in §§ 1, 2 with $m = n+1$.

For the hypersurface V^n at a point P, the normal space in the tangent space of the manifold V^{n+1} is of dimension one, and there is only one second fundamental form Φ_{n+1}. For simplicity we shall write $b_{\alpha\beta}$ for $b^{n+1}{}_{\alpha\beta}$ in equation (2.7), and therefore we have

$$\Phi_{n+1} = b_{\alpha\beta}\omega^{\alpha}\omega^{\beta}. \tag{3.1}$$

The element of area of the hypersurface V^n at the point P is

$$dA = g^{\frac{1}{2}}dx^1 \wedge \cdots \wedge dx^n. \tag{3.2}$$

For the unit normal vector e_{n+1} of the hypersurface V^n at the point P, we have

$$a_{ij}e_{n+1}{}^i e_{n+1}{}^j = 1, \tag{3.3}$$

$$a_{ij}e_{n+1}{}^i y^j{}_{,\alpha} = 0 \qquad (\alpha = 1, \cdots, n). \tag{3.4}$$

From the definition of the vector product of n tangent vectors of the manifold V^{n+1} at a point, it follows that $e_1 \times \cdots \times e_n$ is orthogonal to each of the n vectors $e_1, \cdots, e_n$. Therefore we can write

$$e_1 \times \cdots \times e_n = ce_{n+1}, \tag{3.5}$$

where c is a function of the x's.

Consider the two matrices

$$\phi = (\phi_j{}^{\alpha}), \qquad \psi = (\psi_{\alpha}{}^j); \tag{3.6}$$

where

$$\phi_j{}^{\alpha} = a_{ij}e_{\alpha}{}^i, \qquad \psi_{\alpha}{}^j = e_{\alpha}{}^j, \tag{3.7}$$

the superscript of the element $\phi_j{}^{\alpha}$ or $\psi_{\alpha}{}^j$ indicating the row to which the element belongs, and the subscript indicating the column. From equations (1.2) and (3.5), it is easily seen that

$$ce_{n+1}{}^r = (-1)^{n+r+1}A^r a^{-\frac{1}{2}} \qquad (r = 1, \cdots, n+1), \tag{3.8}$$

where A^r is the determinant of the n-th order obtained by deleting the r-th column from the matrix ϕ. Substitution of equation (3.8) in equation (3.3) gives

$$c^2 = A, \tag{3.9}$$

where

$$A = \begin{vmatrix} A^1 & -A^2 & \cdots & (-1)^n A^{n+1} \\ e_1{}^1 & e_1{}^2 & \cdots & e_1{}^{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ e_n{}^1 & e_n{}^2 & \cdots & e_n{}^{n+1} \end{vmatrix}, \tag{3.10}$$

which is equal to the sum of the products of the corresponding determinants of the n-th order of the two matrices (3.6). By an elementary theorem on determinants (see, for instance, [5], p. 102), from equation (1.5) it follows immediately that

$$A = |\phi_j{}^\alpha \psi_\beta{}^j| = 1. \tag{3.11}$$

For an orientable hypersurface V^n we can choose the direction of the unit normal vector e_{n+1} such that we have, in consequence of equations (3.5), (3.9) and (3.11),

$$e_1 \times \cdots \times e_n = e_{n+1}. \tag{3.12}$$

Taking the scalar product of the vector e_{n+1} with each side of equation (3.12) and using equation (1.4), we can easily obtain

$$|e_1, \cdots, e_{n+1}| = 1/a^{\frac{1}{2}}. \tag{3.13}$$

Similarly, by choosing the orientation of the frame $Pe_1 \cdots e_n$ in the tangent space of an orientable hypersurface V^n at a point P to be that of the frame $PY_{,1} \cdots Y_{,n}$, we can have

$$Y_{,1} \times \cdots \times Y_{,n} = g^{\frac{1}{2}} e_{n+1}. \tag{3.14}$$

By taking the generalized covariant derivative of each side of equations (2.2), (3.3) and (3.4) with respect to the x's, we can easily obtain

$$y^i{}_{;\alpha\beta} = \Omega_{\alpha\beta} e_{n+1}{}^i, \tag{3.15}$$

$$\Omega_{\alpha\beta} = a_{ij} y^i{}_{;\alpha\beta} e_{n+1}{}^j, \tag{3.16}$$

$$e_{n+1}{}^j{}_{;\alpha} = -\Omega_{\alpha\gamma} g^{\gamma\delta} y^j{}_{,\delta}, \tag{3.17}$$

where $(g^{\alpha\beta})$ is the inverse matrix of $(g_{\alpha\beta})$, and $\Omega_{\alpha\beta}$ $(= \Omega_{\beta\alpha})$, different from the coefficients $b_{\alpha\beta}$ in equation (3.1), are the coefficients of the second fundamental form of the hypersurface V^n at the point P with respect to $g_{\alpha\beta}$ and the x's. Taking the generalized covariant derivative of each side of equation (3.17), we obtain

$$e_{n+1}{}^j{}_{;\alpha\beta} = -\Omega_{\alpha\gamma;\beta} g^{\gamma\delta} y^j{}_{,\delta} - \Omega_{\alpha\gamma} g^{\gamma\delta} y^j{}_{;\delta\beta}, \tag{3.18}$$

and therefore,

$$\begin{aligned} e_{n+1}{}^j{}_{;\alpha\beta} - e_{n+1}{}^j{}_{;\beta\alpha} = {} & g^{\gamma\delta} y^j{}_{,\delta} (\Omega_{\beta\gamma;\alpha} - \Omega_{\alpha\gamma;\beta}) \\ & + g^{\gamma\delta} (\Omega_{\beta\gamma} y^j{}_{;\delta\alpha} - \Omega_{\alpha\gamma} y^j{}_{;\delta\beta}). \end{aligned} \tag{3.19}$$

The equations of Gauss and Codazzi for the hypersurface V^n (see, for instance, [5], p. 149) are

$$R_{\alpha\beta\gamma\delta} = (\Omega_{\alpha\gamma}\Omega_{\beta\delta} - \Omega_{\alpha\delta}\Omega_{\beta\gamma}) + \bar{R}_{hijk}y^h{}_{,\alpha}y^i{}_{,\beta}y^j{}_{,\gamma}y^k{}_{,\delta}, \tag{3.20}$$

$$\Omega_{\alpha\beta,\gamma} - \Omega_{\alpha\gamma,\beta} = -\bar{R}_{hijk}e_{n+1}{}^h y^i{}_{,\alpha}y^j{}_{,\beta}y^k{}_{,\gamma}, \tag{3.21}$$

where $R_{\alpha\beta\gamma\delta}$ and $\bar{R}_{hijk}$ are Riemann symbols formed with the tensors $g_{\alpha\beta}$ and a_{ij} respectively.

In particular, if the manifold V^{n+1} is of constant Riemannian curvature K, from the definition of the Riemannian curvature K it follows that

$$\bar{R}_{hijk} = K(a_{hj}a_{ik} - a_{hk}a_{ji}), \tag{3.22}$$

and therefore equations (3.20) and (3.21) can be reduced to

$$R_{\alpha\beta\gamma\delta} = (\Omega_{\alpha\gamma}\Omega_{\beta\delta} - \Omega_{\alpha\delta}\Omega_{\beta\gamma}) + K(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}), \tag{3.23}$$

$$\Omega_{\alpha\beta,\gamma} - \Omega_{\alpha\gamma,\beta} = 0. \tag{3.24}$$

Furthermore, substitution of equations (3.15), (3.20) and (3.24) in equation (3.19) and use of $\Omega_{\alpha\beta} = \Omega_{\beta\alpha}$ give immediately

$$e_{n+1\,;\alpha\beta} - e_{n+1\,;\beta\alpha} = \psi_{\alpha\beta}e_{n+1}, \tag{3.25}$$

where

$$\psi_{\alpha\beta} = g^{\gamma\delta}(R_{\beta\alpha\gamma\delta} - \bar{R}_{hijk}y^h{}_{,\beta}y^i{}_{,\alpha}y^j{}_{,\gamma}y^k{}_{,\delta}) = 0. \tag{3.26}$$

Thus we obtain

$$d^2e_{n+1} = \tfrac{1}{2}\psi_{\alpha\beta}(dx^\beta \wedge dx^\alpha)e_{n+1} = 0. \tag{3.27}$$

From equations (2.6) and (2.7) it is easily seen that the n principal curvatures $k_1, \cdots, k_n$ of the hypersurface V^n at a point P are roots of the determinant equation

$$|\, b_{\alpha\beta} - k\delta_{\alpha\beta}\,| = 0. \tag{3.28}$$

In other words, $k_1, \cdots, k_n$ are the characteristic roots of the matrix $(b_{\alpha\beta})$. The α-th mean curvature M_α of the hypersurface V^n at the point P is defined by

$$C_\alpha{}^n M_\alpha = S_\alpha \qquad (\alpha = 1, \cdots, n), \tag{3.29}$$

where S_α is the α-th elementary symmetric function of $k_1, \cdots, k_n$, and $C_\alpha{}^n$ is a binomial coefficient.

A principal minor of the matrix $(b_{\alpha\beta})$ is a minor whose diagonal is a part of the main diagonal of the matrix $(b_{\alpha\beta})$. From a theorem in linear algebra (see, for instance, [16], Chapter VII) it is known that the α-th elementary symmetric function of the characteristic roots of the matrix $(b_{\alpha\beta})$ is equal to the sum of all α-rowed principal minors of the matrix $(b_{\alpha\beta})$.

Therefore $C_{\alpha}{}^{n}M_{\alpha}$ is equal to the sum of all α-rowed principal minors of the matrix $(b_{\alpha\beta})$. In particular, we have

$$nM_1 = \sum_{\alpha=1}^{n} b_{\alpha\alpha}, \tag{3.30}$$

$$M_n = |\, b_{\alpha\beta} \,| = b, \tag{3.31}$$

$$nM_{n-1} = \sum_{\alpha=1}^{n} B^{\alpha\alpha}, \tag{3.32}$$

where $B^{\alpha\alpha}$ is the cofactor of $b_{\alpha\alpha}$ in the determinant b.

We shall introduce the combined operator $\otimes$ of the vector product $\times$ and the exterior product $\wedge$, as defined in the paper [12] for the case where the manifold V^{n+1} is a Euclidean space.

Let $A^1, \cdots, A^n$ be n tangent vectors of the manifold V^{n+1} at a point P, and suppose that the components of each vector A^{α} $(\alpha = 1, \cdots, n)$ in the local coordinates $y^1, \cdots, y^{n+1}$ of the point P are differentiable functions of the n variables $x^1, \cdots, x^n$. Combining the vector product of vectors and the exterior product of differentials, we can define the vector

$$\begin{aligned} &A^1 \otimes \cdots \otimes A^j \otimes dA^{j+1} \otimes \cdots \otimes dA^n \\ &= (A^1 \times \cdots \times A^j \times A^{j+1}{}_{;i_{j+1}} \times \cdots \times A^n{}_{;i_n})\, dx^{i_{j+1}} \wedge \cdots \wedge dx^{i_n}, \\ &\qquad\qquad (j = 1, \cdots, n). \end{aligned} \tag{3.33}$$

It is obvious that the vector (3.33) is independent of the order of the vectors $dA^{j+1}, \cdots, dA^n$.

From equations (1.6), (2.3), (3.2), (3.12) and (3.14) it is easy to obtain

$$\underbrace{dY \otimes \cdots \otimes dY}_{n} = n!(\omega^1 \wedge \cdots \wedge \omega^n)\, e_{n+1}, \tag{3.34}$$

$$\begin{aligned} \underbrace{dY \otimes \cdots \otimes dY}_{n} &= Y_{,\alpha_1} dx^{\alpha_1} \otimes \cdots \otimes Y_{,\alpha_n} dx^{\alpha_n} \\ &= n!(Y_{,1} \times \cdots \times Y_{,n})\, dx^1 \wedge \cdots \wedge dx^n \\ &= n! e_{n+1} dA. \end{aligned} \tag{3.35}$$

Comparison of equation (3.34) with equation (3.35) yields immediately

$$dA = \omega^1 \wedge \cdots \wedge \omega^n. \tag{3.36}$$

Similarly, by means of equations (1.6), (1.7), (1.8), (2.3) and (2.5), we have

$$\underbrace{dY \otimes \cdots \otimes dY}_{n-\alpha} \otimes \underbrace{de_{n+1} \otimes \cdots \otimes de_{n+1}}_{\alpha}$$

$$= (-1)^{\alpha}(n-\alpha)!\,\alpha! \sum_{\substack{\beta_1 < \cdots < \beta_{n-\alpha} \\ \beta_{n-\alpha+1} < \cdots < \beta_n}} (\omega^{\beta_1} e_{\beta_1} \otimes \cdots \otimes \omega^{\beta_{n-\alpha}} e_{\beta_{n-\alpha}}$$

$$\otimes b_{\beta_{n-\alpha+1},\gamma_1} \omega^{\gamma_1} e_{\beta_{n-\alpha+1}} \otimes \cdots \otimes b_{\beta_n,\gamma_\alpha} \omega^{\gamma_\alpha} e_{\beta_n}).$$

It is easily seen that in the last expression we need only to consider the terms, for which $\gamma_1, \cdots, \gamma_\alpha$ take the values $\beta_{n-\alpha+1}, \cdots, \beta_n$, and therefore the subscripts of the b's will contain none of the values $\beta_1, \cdots, \beta_{n-\alpha}$; as all other terms are zero. Making use of equations (3.12), (3.36) and the elementary identity $C_\alpha^n = n!/[(n-\alpha)!\alpha!]$, we thus obtain

$$(3.37) \qquad \underbrace{dY \otimes \cdots \otimes dY}_{n-\alpha} \otimes \underbrace{de_{n+1} \otimes \cdots \otimes de_{n+1}}_{\alpha}$$

$$= (-1)^{\alpha}(n-\alpha)!\,\alpha!(e_1 \times \cdots \times e_n)(\omega^1 \wedge \cdots \wedge \omega^n)$$

[sum of all α-rowed principal minors of the matrix $(b_{\alpha\beta})$]

$$= (-1)^{\alpha} n!\, M_\alpha e_{n+1}\, dA.$$

A closed hypersurface V^n in a Riemannian manifold V^{n+1} of dimension $n+1 \geqq 3$ is called a Riemann n-sphere if every point of the hypersurface V^n is umbilical. Concerning umbilical points the following lemma is well known.

LEMMA 3.1. *A point P of a hypersurface V^n imbedded in a Riemannian manifold V^{n+1} of dimension $n+1 \geqq 3$ is umbilical if $k_1 = \cdots = k_n$ at the point P.*

Proof. It is well known that the principal curvatures $k_1, \cdots, k_n$ of the hypersurface V^n at the point P are the extremal values of the quantity k defined by

$$k = (\Omega_{\alpha\beta} q^\alpha q^\beta)/(g_{\alpha\beta} q^\alpha q^\beta)$$

at the point P for an arbitrary tangent vector q of the hypersurface V^n with contravariant components q^α. Thus the assumption that $k_1 = \cdots = k_n$ at the point P implies that the quantity k is independent of the vector q, so that $\Omega_{\alpha\beta} = c g_{\alpha\beta}$ for all α and β at the point P, where c is a scalar invariant. Hence the point P of the hypersurface V^n is umbilical (see, for instance, [5], p. 179).

Now consider, for example,[4] a three-dimensional Riemannian space V^3 with the fundamental form

[4] The authors are indebted to the referee for suggesting this example.

$$ds^2 = (\sum_{i=1}^{3} y_i^2)^c \sum_{j=1}^{3} dy_j^2,$$

where c is a constant $\geqq 2$. A straightforward calculation shows that the space V^3, excluding the origin $(0,0,0)$, is not of constant curvature and that every point of the surface in the space V^3 given by the parametric equations

$$y^1 = \sin x^1 \cos x^2, \qquad y^2 = \sin x^1 \sin x^2, \qquad y^3 = \cos x^1$$

is umbilical.

4. Lemmas. In this section we shall state two lemmas, which will be needed for later development. The proofs of the lemmas will be omitted here, but can be found in [7], pp. 52, 104, 105.

LEMMA 4.1. *Let S_r be the r-th elementary symmetric function of n real nonzero numbers $k_1, \cdots k_n$, and suppose that $M_0 = 1$ and $M_r = S_r/C_r^n$ for $r = 1, \cdots, n$. Then*

$$M_{r-1}M_{r+1} - M_r^2 \leqq 0 \qquad (r = 1, \cdots, n-1), \tag{4.1}$$

where the equality for any value of r implies that $k_1 = \cdots = k_n$.

LEMMA 4.2. *Let $k_1, \cdots, k_n$, M_0 and M_r be defined as in Lemma 4.1. If $M_1, \cdots, M_s$, $1 \leqq s \leqq n$, are positive, then*

$$M_1 \geqq M_2^{\frac{1}{2}} \geqq M_3^{1/3} \geqq \cdots \geqq M_s^{1/s} \qquad (1 \leqq s \leqq n), \tag{4.2}$$

where the equality at any stage implies that $k_1 = \cdots = k_n$.

5. Riemann n-spheres on Riemannian manifolds of constant curvature. Let O be a point on a Riemannian manifold V^{n+1} of dimension $n+1$, and U a sufficiently small neighborhood of the point O on the manifold V^{n+1} such that in the neighborhood U there exists a unique geodesic ρ joining the point O to every point P of the neighborhood U. Let $g_1, \cdots, g_{n+1}$ be $n+1$ mutually orthogonal geodesics of the manifold V^{n+1} through the point O. Then the normal coordinates $y^1, \cdots, y^{n+1}$ of Riemann of the point P with respect to the geodesic frame $Og_1 \cdots g_{n+1}$ are defined by $y^i = s\cos(\rho, g_i)$, where (ρ, g_i) is the angle between the geodesic ρ and the geodesic g_i at the point O, and s is the arc length from the point O to the point P along the geodesic ρ. It is obvious that $\sum_{i=1}^{n+1} \cos^2(\rho, g_i) = 1$. If on the manifold V^{n+1} there exists a unique geodesic arc with minimal length joining a fixed point

O to every point P, then we can have a normal coordinate system of Riemann at the fixed point O covering the whole manifold V^{n+1}.

THEOREM 5.1. *Let V^{n+1} be a Riemannian manifold of dimension $n+1 \geqq 3$ and constant Riemannian curvature K such that there is a normal coordinate system S of Riemann at a fixed point O covering the whole manifold V^{n+1}. Then for a closed orientable hypersurface V^n of class C^3 imbedded in the manifold V^{n+1}*

$$\int_{V^n} M_{\alpha-1}\, dA + \int_{V^n} M_\alpha p\, dA = 0 \qquad (\alpha = 1, \cdots, n), \tag{5.1}$$

where $|M_1|$ is sufficiently large at least at some points of the hypersurface V^n, and p is the scalar product of the unit normal vector e_{n+1} of the hypersurface V^n at a point P and the position vector Y of the point P with respect to the coordinate system S.

Proof. We first consider an orientable hypersurface V^n of class C^3 with boundary V^{n-1} of dimension $n-1 \geqq 1$ imbedded in the Riemannian manifold V^{n+1}, and shall use the results of the preceding sections with the same notations except that here Y is the position vector of a point P on the hypersurface V^n with respect to the fixed normal coordinate system S of Riemann.

Applying the ordinary rules for differentiation of determinants and recalling that a can be regarded as a constant in generalized covariant differentiation, we obtain the differential form

$$\begin{aligned}
(5.2)\quad & d[(-1)^n a^{\lambda} \,|\, \underbrace{dY, \cdots, dY}_{n-\alpha}, Y, e_{n+1}, \underbrace{de_{n+1}, \cdots, de_{n+1}}_{\alpha-1} \,|\,] \\
&= \sum_{\beta=1}^{n-\alpha} [(-1)^n a^{\lambda} \,|\, \underbrace{\overbrace{dY, \cdots, dY}^{\beta-1}, d^2Y, dY, \cdots, dY}_{n-\alpha}, Y, e_{n+1}, \underbrace{de_{n+1}, \cdots, de_{n+1}}_{\alpha-1} \,|\,] \\
&\quad + (-1)^{\alpha-1} a^{\lambda} \,|\, e_{n+1}, \underbrace{dY, \cdots, dY}_{n-\alpha+1}, \underbrace{de_{n+1}, \cdots, de_{n+1}}_{\alpha-1} \,| \\
&\quad + (-1)^{\alpha} a^{\lambda} \,|\, Y, \underbrace{dY, \cdots, dY}_{n-\alpha}, \underbrace{de_{n+1}, \cdots, de_{n+1}}_{\alpha} \,| \\
&+ \sum_{\beta=1}^{\alpha-1} [(-1)^n a^{\lambda} \,|\, \underbrace{dY, \cdots, dY}_{n-\alpha}, Y, e_{n+1}, \underbrace{\overbrace{de_{n+1}, \cdots, de_{n+1}}^{\beta-1}, d^2e_{n+1}, de_{n+1}, \cdots, de_{n+1}}_{\alpha-1} \,|\,],
\end{aligned}$$

where $\alpha = 1, \cdots, n$. By means of the equation $d^2Y = 0$ mentioned in §1

and equation (3.27), it follows immediately that the first and fourth members on the right side of equation (5.2) vanish. We thus obtain, in consequence of equations (1.4) and (3.37),

$$
\begin{aligned}
(5.3)\quad & d[(-1)^n a^{\lambda} \mid \underbrace{dY, \cdots, dY}_{n-\alpha}, Y, e_{n+1}, \underbrace{de_{n+1}, \cdots, de_{n+1}}_{\alpha-1} \mid] \\
&= (-1)^{n-\alpha+1} e_{n+1} \cdot (\underbrace{dY \otimes \cdots \otimes dY}_{n-\alpha+1} \otimes \underbrace{de_{n+1} \otimes \cdots \otimes de_{n+1}}_{\alpha-1}) \\
&\quad + (-1)^{n-\alpha} Y \cdot (\underbrace{dY \otimes \cdots \otimes dY}_{n-\alpha} \otimes \underbrace{de_{n+1} \otimes \cdots \otimes de_{n+1}}_{\alpha}) \\
&= (-1)^n n! (M_{\alpha-1}\, dA + M_\alpha p\, dA),
\end{aligned}
$$

where $p = Y \cdot e_{n+1}$. Integrating equation (5.3) over the hypersurface V^n and applying Stokes' Theorem to the left side of the equation, we arrive at the integral formulas

$$
\begin{aligned}
(5.4)\quad & \int_{V^n} M_{\alpha-1}\, dA + \int_{V^n} M_\alpha p\, dA \\
&= (1/n!) \int_{V^{n-1}} a^{\lambda} \mid \underbrace{dY, \cdots, dY}_{n-\alpha}, Y, e_{n+1}, \underbrace{de_{n+1}, \cdots, de_{n+1}}_{\alpha-1} \mid, \\
&\qquad\qquad (\alpha = 1, \cdots, n).
\end{aligned}
$$

In particular, when V^n is closed, the right side of equation (5.4) vanishes, and hence we have the formulas (5.1).

THEOREM 5.2. *Let V^n $(n \geqq 2)$ be a hypersurface on a Riemannian manifold V^{n+1} satisfying the conditions of Theorem 5.1. Suppose that there exists an integer s, $1 \leqq s \leqq n-1$, such that $M_s > 0$, and either $p \leqq -M_{s-1}/M_s$ or $p \geqq -M_{s-1}/M_s$ at all points of the hypersurface V^n. Then V^n is a Riemann n-sphere.*

Proof. Since $M_s > 0$, the conditions $p \leqq -M_{s-1}/M_s$ and $p \geqq -M_{s-1}/M_s$ are respectively equivalent to $M_s p + M_{s-1} \leqq 0$ and $M_s p + M_{s-1} \geqq 0$. Equation (5.1) for $\alpha = s$, together with either of these two inequalities, implies that $p = -M_{s-1}/M_s$. Substituting this value of p in equation (5.1) for $\alpha = s+1$, we obtain

$$
(5.5)\qquad \int_{V^n} (1/M_s)\,(M_s^2 - M_{s-1}M_{s+1})\, dA = 0.
$$

Due to the inequality (4.1) for $r = s$, the integrand on the left side of equation

(5.5) is nonnegative, and therefore equation (5.5) holds when and only when

$$M_s^2 - M_{s-1}M_{s+1} = 0. \tag{5.6}$$

From Lemma 4.1 it follows immediately that $k_1 = \cdots = k_n$ at all points of the hyersurface V^n, and by Lemma 3.1 we thus prove Theorem 5.2.

THEOREM 5.3. *Let V^n $(n \geqq 2)$ be a hypersurface on a Riemannian manifold V^{n+1} satisfying the conditions of Theorem 5.1. Suppose that there exists an integer s, $1 \leqq s \leqq n$, such that at all points of the hypersurface V^n the function p is of the same sign, $M_i > 0$ for $i = 1, \cdots, s$, and M_s is constant. Then the hypersurface V^n is a Riemann n-sphere.*

Proof. Case 1. $s < n$. By the inequalities (4.1) for $r = 1, \cdots, s$ and the assumption that $M_i > 0$ for $i = 1, \cdots, s$, we obtain

$$M_1/M_0 \geqq M_2/M_1 \geqq \cdots \geqq M_{s+1}/M_s,$$

and in particular

$$M_1 M_s \geqq M_{s+1}, \tag{5.7}$$

where the equality implies that $k_1 = \cdots = k_n$. From equation (5.1) for $\alpha = 1$ and the assumptions that $M_1 > 0$ and p is of the same sign at all points of the hypersurface V^n, it follows that p must be negative. Multiplying both sides of the inequality (5.7) by p, integrating over the hypersurface V^n, and applying equation (5.1) for $\alpha = 1$ and $\alpha = s+1$, we can readily obtain, in consequence of the assumption that M_s is constant,

$$-M_s \int_{V^n} dA = \int_{V^n} M_1 M_s p \, dA \leqq \int_{V^n} M_{s+1} p \, dA = -M_s \int_{V^n} dA,$$

from which it follows that

$$\int_{V^n} (M_1 M_s - M_{s+1}) p \, dA = 0. \tag{5.8}$$

Due to the inequality (5.7) the integrand on the left side of equation (5.8) is nonpositive, and therefore $M_1 M_s - M_{s+1} = 0$. From Lemma 4.1 it follows that $k_1 = \cdots = k_n$ at all points of the hypersurface V^n, and by Lemma 3.1 Theorem 5.3 for $s < n$ is therefore proved.

Case 2. $s = n$. By using the assumption that $M_i > 0$ for $i = 1, \cdots, n$, from Lemma 4.2 we have the inequalities

$$M_1 \geqq M_2^{1/2} \geqq \cdots \geqq M_{n-1}^{1/(n-1)} \geqq M_n^{1/n} = c, \tag{5.9}$$

where c is a positive constant. By means of equation (5.1) for $\alpha = n$ and

the inequalities (5.9) we obtain

$$\int_{V^n} M_n p\, dA = -\int_{V^n} M_{n-1}\, dA \leqq -c^{n-1}\int_{V^n} dA. \tag{5.10}$$

On the other hand, making use of equation (5.1) for $\alpha = 1$, the inequalities (5.9) and the fact that $p < 0$, we have

$$\begin{aligned}\int_{V^n} M_n p\, dA &= c^{n-1}\int_{V^n} M_n^{1/n} p\, dA \\ &\geqq c^{n-1}\int_{V^n} M_1 p\, dA = -c^{n-1}\int_{V^n} dA.\end{aligned} \tag{5.11}$$

Combination of (5.10) and (5.11) yields immediately

$$\int_{V^n} (M_n^{1/n} - M_1) p\, dA = 0. \tag{5.12}$$

Due to the inequalities (5.9), the integrand on the left side of equation (5.12) is nonnegative, and therefore $M_n^{1/n} = M_1$, which by Lemma 4.2 implies that $k_1 = \cdots = k_n$ at all points of the hypersurface V^n. Hence the proof of Theorem 5.3 is complete by Lemma 3.1.

In the case where V^{n+1} is a Euclidean space and V^n is a convex hypersurface of class C^2 instead of C^3; the formulas (5.1) were obtained by Minkowski [15] for $n = 2$ and by Kubota [13] for a general n, Theorem 5.2 with further restrictions that $s = 1$ and the equality holds in the last conditions was obtained by Grotemeyer [6] for $n = 2$ and by Süss [18] for a general n, and Theorem 5.3 was obtained by Liebmann [14] for $n = 2$ and by Süss [17] for a general n.

By using different methods, Hsiung [11, 10] obtained the formulas (5.1) for $\alpha = 1, n$ and some special cases of Theorems 5.2 and 5.3; and also the formulas (5.1) for $\alpha = 1, \cdots, n$, together with Theorems 5.2 and 5.3, in the case where V^{n+1} is a Euclidean space and the hypersurface V^n is of class C^2 instead of C^3. The proofs of Theorems 5.2 and 5.3 here are essentially the same as those given by Hsiung in the second paper just mentioned.

6. Riemann n-spheres on Riemannian manifolds of dimension $n+1 \geqq 4$.

THEOREM 6.1. *Let V^{n+1} be a Riemannian manifold of dimension $n+1 \geqq 3$ such that there is a normal coordinate system S of Riemann at a fixed point O covering the whole manifold V^{n+1}. Then for a closed orientable hypersurface V^n of class C^3 imbedded in the manifold V^{n+1}*

$$\int_{V^n} M_{\alpha-1}\, dA + \int_{V^n} M_\alpha p\, dA = 0, \tag{6.1}$$

where $|M_1|$ *is sufficiently large at least at some points of the hypersurface* V^n, α *is an odd integer less than or equal to* n, *and* p *is defined as in Theorem* 5.1.

Proof. The proof of Theorem 6.1 is exactly the same as that of Theorem 5.1, except that this time α is an odd integer and the vanishing of the fourth member on the right side of equation (5.2) is due to the pairwise cancellation of its terms.

By using a different method, Hsiung [11] obtained the formula (6.1) for $\alpha = 1$ in the case where the hypersurface V^n is of class C^2 instead of C^3.

THEOREM 6.2. *Let* V^n $(n \geqq 3)$ *be a hypersurface on a Riemannian manifold* V^{n+1} *satisfying the conditions of Theorem* 6.1. *Suppose that there exists an odd integer* s, $1 < s \leqq n$, *such that* $M_i > 0$ *for* $i = s$, $s-1$, $s-2$, *and either* $p \leqq -M_{s-1}/M_s$ *or* $p \geqq -M_{s-1}/M_s$ *at all points of the hypersurface* V^n. *Then* V^n *is a Riemann* n*-sphere.*

Proof. By the same argument as in the proof of Theorem 5.2, equation (6.1) for $\alpha = s$, together with either of the two inequalities $M_s p + M_{s-1} \leqq 0$ and $M_s p + M_{s-1} \geqq 0$, implies that $p = -M_{s-1}/M_s$. Substituting this value of p in equation (6.1) for $\alpha = s-2$, which is an odd integer by assumption, we obtain

$$\int_{V^n} (1/M_s)\,(M_{s-3}M_s - M_{s-2}M_{s-1})\, dA = 0. \tag{6.2}$$

Since M_{s-1} and M_{s-2} are positive, from inequalities (4.1) for $r = s-1$ and $r = s-2$, we obtain

$$M_{s-3}/M_{s-2} \leqq M_{s-2}/M_{s-1} \leqq M_{s-1}/M_s, \tag{6.3}$$

from which it follows that $M_{s-3}M_s - M_{s-2}M_{s-1} \leqq 0$. Thus the integrand on the left side of equation (6.2) is nonpositive, and the equality in (6.3) holds. From Lemma 4.1 it follows immediately that $k_1 = \cdots = k_n$ at all points of the hypersurface V^n, and by Lemma 3.1 Theorem 6.2 is therefore proved.

THEOREM 6.3. *Let* V^n $(n \geqq 3)$ *be a hypersurface on a Riemannian manifold* V^{n+1} *satisfying the conditions of Theorem* 6.1. *Suppose that there exists an odd integer* s, $1 < s \leqq n$, *such that at all points of the hypersurface* V^n *the function* p *is of the same sign,* $M_i > 0$ *for* $i = 1, \cdots, s-1$, *and either* M_{s-1} *or* M_s *is constant. Then* V^n *is a Riemann* n*-sphere.*

Proof. *Case* 1. M_{s-1} is constant. Multiplying both sides of equation (6.1) for $\alpha = 1$ by M_{s-1} and subtracting the resulting equation from equation (6.1) for $\alpha = s$, we obtain

$$\int_{V^n} (M_s - M_1 M_{s-1}) p \, dA = 0. \tag{6.4}$$

Since $M_1, M_2, \cdots, M_{s-1}$ are assumed to be positive, the inequalities (4.1) for $r = 1, \cdots, s-1$ can be written together as

$$M_1 \geqq M_2/M_1 \geqq M_3/M_2 \geqq \cdots \geqq M_s/M_{s-1}, \tag{6.5}$$

from which it follows immediately that $M_s - M_1 M_{s-1} \leqq 0$. On the other hand, by using equation (6.1) for $\alpha = 1$ and the assumptions that $M_1 > 0$ and p is of the same sign at all points of the hypersurface V^n, it is easily seen that p is negative. Thus the integrand on the left side of equation (6.4) is nonnegative, and the equality in (6.5) holds. From Lemma 4.1 it follows immediately that $k_1 = \cdots = k_n$ at all points of the hypersurface V^n, and therefore by Lemma 3.1 V^n is a Riemann n-sphere.

Case 2. M_s is constant. Since s is an odd integer and $M_i > 0$ for $i = 1, \cdots, s-1$, from Lemma 4.2 we have the inequalities

$$M_1 \geqq M_2^{1/2} \geqq \cdots \geqq M_{s-1}^{1/(s-1)} \geqq M_s^{1/s} = c, \tag{6.6}$$

where c is a constant. By means of equation (6.1) for $\alpha = s$ and the inequalities (6.6) we obtain

$$\int_{V^n} M_s p \, dA = -\int_{V^n} M_{s-1} dA \leqq -c^{s-1} \int_{V^n} dA. \tag{6.7}$$

On the other hand, making use of equation (6.1) for $\alpha = 1$, the inequalities (6.6) and the fact that $p < 0$, we have

$$\begin{aligned} \int_{V^n} M_s p \, dA &= c^{s-1} \int_{V^n} M_s^{1/s} p \, dA \\ &\geqq c^{s-1} \int_{V^n} M_1 p \, dA = -c^{s-1} \int_{V^n} dA. \end{aligned} \tag{6.8}$$

Combination of (6.7) and (6.8) yields immediately

$$\int_{V^n} (M_s^{1/s} - M_1) p \, dA = 0. \tag{6.9}$$

Due to the inequalities (6.6), the integrand on the left side of equation (6.9) is nonnegative, and therefore $M_s^{1/s} = M_1$, which by Lemma 4.2 implies that

$k_1 = \cdots = k_n$ at all points of the hypersurface V^n. Using Lemma 3.1 we therefore complete the proof of Theorem 6.3.

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REPRESENTATIONS FOR COMPLETELY CONVEX FUNCTIONS.*

By R. P. Boas, Jr.

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1. Introduction. Let $f(x)$ be a completely convex function on $[0,1]$, that is, an infinitely differentiable function such that $(-1)^n f^{(2n)}(x) \geqq 0$. Such a function is necesarily the restriction to $[0,1]$ of an entire function of exponential type π [1], [3], [4], [5]. The most familiar completely convex function is $\sin \pi x$. Wintner [6] recently established another necessary condition for completely convex functions: if f is completely convex on $[0,1]$, it can be written in the form $f(x) = q(\cos \pi x)$, where $q(z) = (1-z^2)^{\frac{1}{2}} p(z)$, $p(z) = \sum_{r=0}^{\infty} \mu_r z^r$, $\mu_r = \int_{-1}^{1} u^r \, d\alpha(u)$, and $\alpha(u)$ is nondecreasing. However, this condition is not sufficient: as Wintner pointed out, the simplest choices of α (except $\alpha(u) = \operatorname{sgn} u$) fail to lead to completely convex f.

I shall obtain a necessary and sufficient condition for an f of Wintner's form to be completely convex. In the light of this condition, it is not surprising that simple choices of α fail to generate completely convex functions, since it turns out that α must be, except for a possible jump at 0, absolutely continuous, with a derivative $F(u)$ of the form

$$\phi\{\tfrac{1}{\pi} \log[u^{-1}(-1 + (1-u^2)^{\frac{1}{2}})]\}(1-u^2)^{-\frac{1}{2}}, \qquad u < 0;$$

$$\psi\{\tfrac{1}{\pi} \log[u^{-1}(1 + (1-u^2)^{\frac{1}{2}})]\}(1-u^2)^{-\frac{1}{2}}, \qquad u > 0,$$

where $\phi(t)$ and $\psi(t)$ are themselves entire functions, of the form

$$\sum_{n=0}^{\infty} a_n t^{2n}/(2n)!$$

with $a_n \geqq 0$ and $\sum \pi^{-2n} a_n$ convergent. A perhaps simpler characterization of completely convex functions on $[0,1]$ is by means of the integral representation

$$(1) \qquad f(x) = \sin \pi x \cdot \{c + \int_0^{\infty} (\cosh \pi t + \cos \pi x)^{-1} \phi(t)\, dt + \int_0^{\infty} (\cosh \pi t - \cos \pi x)^{-1} \psi(t)\, dt\},$$

where $c \geqq 0$ and ϕ and ψ are as specified in the preceding sentence.

I am indebted to D. V. Widder for helpful discussions.

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2. The integral representation. If $f(x)$ is completely convex in $[0,1]$, there is a largest number c such that $f(x) - c\sin\pi x$ is also completely convex function $g(x)$. Then $f(x) = c\sin\pi x + g(x)$, and $g(x)$ is a minimal completely convex function; that is, $g(x)$ is completely convex, and no function $g(x) - c\sin\pi x$, $c > 0$, is completely convex. The results of the present paper depend on the following known characterization [4] of minimal completely convex functions by their expansions in Lidstone series: $f(x)$ is a minimal completely convex function on $[0,1]$ if and only if

$$f(x) = \sum_{n=0}^{\infty} \{a_n \Lambda_n(x) + b_n \Lambda_n(1-x)\}, \tag{2}$$

where $(-1)^n a_n \geqq 0$, $(-1)^n b_n \geqq 0$ (in fact, $a_n = f^{(2n)}(1)$, $b_n = f^{(2n)}(0)$), and $\Lambda_n(x)$ is the polynomial of degree n which for $0 < x < 1$ has the Fourier series

$$\Lambda_n(x) = 2\pi^{-2n-1}(-1)^n \sum_{k=1}^{\infty} (-1)^{k+1} k^{-2n-1} \sin k\pi x. \tag{3}$$

Suppose that f has the representation (2); it is known that the series converges for all complex values of x, uniformly on compact sets. Since we have [2]

$$-i(-1)^n \Lambda_n(iy) \geqq \tfrac{3}{2} y \pi^{-2n} \tag{4}$$

and since $(-1)^n f^{(2n)}(1) \geqq 0$, the convergence of the first part of (2) implies that

$$\sum |f^{(2n)}(1)| \pi^{-2n} \quad \text{converges} \tag{5}$$

and hence, since $f(1-x)$ is a minimal completely convex function with $f(x)$, that $\sum |f^{(2n)}(0)| \pi^{-2n}$ converges.

Using the Fourier series (3), we now have, for $0 < x < 1$,

$$\begin{aligned} f(x) &= 2\sum_{n=0}^{\infty} (-1)^n f^{(2n)}(1) \pi^{-2n-1} \sum_{k=1}^{\infty} (-1)^{k+1} k^{-2n-1} \sin k\pi x \\ &\quad + 2\sum_{n=0}^{\infty} (-1)^n f^{(2n)}(0) \pi^{-2n-1} \sum_{k=1}^{\infty} k^{-2n-1} \sin k\pi x \\ &= 2\Im\{ \sum_{n=0}^{\infty} (-1)^n f^{(2n)}(1) \sum_{k=1}^{\infty} (-1)^{k+1} (k\pi)^{-2n-1} e^{ik\pi x} \\ &\quad + \sum_{n=0}^{\infty} (-1)^n f^{(2n)}(0) \sum_{k=1}^{\infty} (k\pi)^{-2n-1} e^{ik\pi x} \} \qquad (6) \\ &= 2\Im\{ \sum_{n=0}^{\infty} (-1)^n f^{(2n)}(1)/(2n)! \sum_{k=1}^{\infty} (-1)^{k+1} e^{ik\pi x} \int_0^{\infty} e^{-k\pi t} t^{2n}\, dt \\ &\quad + \sum_{n=0}^{\infty} (-1)^n f^{(2n)}(0)/(2n)! \sum_{k=1}^{\infty} e^{ik\pi x} \int_0^{\infty} e^{-k\pi t} t^{2n}\, dt \}. \end{aligned}$$

Now

$$\sum_{k=1}^{\infty} \int_0^{\infty} e^{-k\pi t} t^{2n}\, dt = \sum_{k=1}^{\infty} (2n)\,!\,(k\pi)^{-2n-1}$$

converges for $n > 0$, so for $n > 0$ the integrals in (6) can be taken outside the sums over k. For $n = 0$, we have by direct computation

$$\sum_{k=1}^{\infty} e^{ik\pi x} \int_0^{\infty} e^{-k\pi t}\, dt = \sum_{k=1}^{\infty} (k\pi)^{-1} e^{ik\pi x} = -\pi^{-1} \log(1 - e^{i\pi x})$$

and

$$\int_0^{\infty} dt \sum_{k=1}^{\infty} e^{ik\pi x - k\pi t} = \int_0^{\infty} (e^{\pi t - i\pi x} - 1)^{-1}\, dt = -\pi^{-1} \log(1 - e^{i\pi x}),$$

so that the change of order of operations is also legitimate for $n = 0$. Hence we have

$$f(x) = 2\Im\{ \sum_{n=0}^{\infty} (-1)^n f^{(2n)}(1)/(2n)\,! \int_0^{\infty} t^{2n} \sum_{k=1}^{\infty} (-1)^{k+1} e^{ik\pi x - k\pi t}\, dt$$
$$+ \sum_{n=0}^{\infty} (-1)^n f^{(2n)}(0)/(2n)\,! \int_0^{\infty} t^{2n} \sum_{k=1}^{\infty} e^{ik\pi x - k\pi t}\, dt\},$$

$$(7) \quad f(x) = 2\Im\{ \sum_{n=0}^{\infty} (-1)^n f^{(2)n}(1)/(2n)\,! \int_0^{\infty} t^{2n} [e^{-\pi(ix-t)} + 1]^{-1}\, dt$$
$$+ \sum_{n=0}^{\infty} (-1)^n f^{(2n)}(0)/(2n)\,! \int_0^{\infty} t^{2n} [e^{-\pi(ix-t)} - 1]^{-1}\, dt\}.$$

Now

$$\sum_{n=0}^{\infty} |f^{(2n)}(1)|/(2n)\,! \int_0^{\infty} e^{-\pi t} t^{2n}\, dt$$

converges by (5). Since $|e^{-\pi(ix-t)} + 1| \geqq e^{\pi t} - 1$,

$$\sum_{n=0}^{\infty} |f^{(2n)}(1)|/(2n)\,! \int_1^{\infty} t^{2n} |e^{-\pi(ix-t)} + 1|^{-1}\, dt$$

converges; and

$$\sum_{n=0}^{\infty} |f^{(2n)}(1)|/(2n)\,! \int_0^{1} t^{2n} |e^{-\pi(ix-t)} + 1|^{-1}\, dt$$

converges (for fixed x) since the integrand is bounded. Thus in the first sum in (7) we can change the order of summation and integration, and similar considerations apply to the second sum. Put

$$\phi(t) = \sum_{n=0}^{\infty} (-1)^n f^{(2n)}(1) t^{2n}/(2n)\,! = \tfrac{1}{2}[f(1+it) + f(1-it)],$$

$$\psi(t) = \sum_{n=0}^{\infty} (-1)^n f^{(2n)}(0) t^{2n}/(2n)\,! = \tfrac{1}{2}[f(it) + f(-it)].$$

Then from (7) we obtain

$$f(x) = 2\Im\{\int_0^\infty \phi(t)[e^{-\pi(ix-t)}+1]^{-1}\,dt + \int_0^\infty \psi(t)[e^{-\pi(ix-t)}-1]^{-1}\,dt\},$$

$$(8)\quad f(x) = \sin\pi x\cdot\{\int_0^\infty \phi(t)[\cosh\pi t+\cos\pi x]^{-1}\,dt + \int_0^\infty \psi(t)[\cosh\pi t-\cos\pi x]^{-1}\,dt\}$$

(which we may recognize as the representation of $f(x)$ by the Poisson integral for the strip $0<x<1$). Thus every minimal completely convex function has the representation (8).

Conversely, if functions $\phi(t)$ and $\psi(t)$ are defined by

$$(9)\qquad \phi(t) = \sum_{n=0}^\infty a_n t^{2n}/(2n)!, \qquad \psi(t) = \sum_{n=0}^\infty b_n t^{2n}/(2n)!,$$

with $a_n \geqq 0$, $b_n \geqq 0$, and $\sum \pi^{-2n}a_n$, $\sum \pi^{-2n}b_n$ convergent, and we define $f(x)$ by (8) for $0<x<1$, all our steps are reversible and lead to

$$f(x) = \sum_{n=0}^\infty \{(-1)^n a_n \Lambda_n(x) + (-1)^n b_n \Lambda_n(1-x)\},$$

which is known to imply [4] that $f(x)$ is a minimal completely convex function.

Since every completely convex function differs from a minimal one by $c\sin\pi x$, we have proved the following theorem.

THEOREM. *A function f defined on $0\leqq x\leqq 1$ is completely convex if and only if it has the form*

$$(10)\quad f(x) = \sin\pi x\cdot\{c+\int_0^\infty \phi(t)[\cosh\pi t+\cos\pi x]^{-1}\,dt + \int_0^\infty \psi(t)[\cosh\pi t-\cos\pi t]^{-1}\,dt\},$$

where $c\geqq 0$, and ϕ and ψ are even entire functions of exponential type π with nonnegative Maclaurin coefficients.

3. Wintner's representation. The representation (10) can be transformed as follows. Expand $\cosh\pi t+\cos\pi x$ in powers of $(\cos\pi x)/\cosh\pi t$, obtaining

$$f(x) = \sin\pi x\cdot\{\int_0^\infty \phi(t)\,dt\sum_{r=0}^\infty (-1)^r\cos^r\pi x(\cosh\pi t)^{-r-1} + \int_0^\infty \psi(t)\,dt\sum_{r=0}^\infty \cos^r\pi x(\cosh\pi t)^{-r-1}\}.$$

Since $\phi(t)$ and $\psi(t)$ are $O(e^{\pi t})$, the two integrals are dominated by

$$\sum_{r=0}^{\infty} |\cos^r \pi x| \int_0^{\infty} e^{-\pi r t}\, dt = \sum_{r=0}^{\infty} (\pi r)^{-1} |\cos^r \pi x|,$$

which converges for $0 < x < 1$. Hence we have

$$f(x) = \sin \pi x \cdot \{ \sum_{r=0}^{\infty} (-1)^r \cos^r \pi x \int_0^{\infty} \phi(t) (\cosh \pi t)^{-r-1}\, dt + \sum_{r=0}^{\infty} \cos^r \pi x \int_0^{\infty} \psi(t) (\cosh \pi t)^{-r-1}\, dt \}.$$

In the first integral put $\cosh \pi t = -u^{-1}$, in the second put $\cosh \pi t = u^{-1}$; we obtain

$$f(x) = \pi^{-1} \sin \pi x \cdot \{ \sum_{r=0}^{\infty} \cos^r \pi x \int_{-1}^{0} \phi\{\tfrac{1}{\pi} \cosh^{-1}(-1/u)\} u^r (1-u^2)^{-\frac{1}{2}}\, du + \sum_{r=0}^{\infty} \cos^r \pi x \int_0^{1} \psi\{\tfrac{1}{\pi} \cosh^{-1}(1/u)\} u^r (1-u^2)^{-\frac{1}{2}}\, du$$

$$= \sin \pi x \cdot \sum_{r=0}^{\infty} \cos^r \pi x \int_{-1}^{1} u^r F(u)\, du,$$

where (in particular) $F(u) \geqq 0$. Thus if we put

$$\mu_r = \int_{-1}^{1} u^r F(u)\, du, \tag{11}$$

$$p(z) = \sum_{r=0}^{\infty} \mu_r z^r,$$

$$q(z) = (1-z^2)^{\frac{1}{2}} p(z),$$

we have $f(x) = q(\cos \pi x)$, which is Wintner's result. However, we have not only this necessary condition, but also the necessary and sufficient condition that f has this form, where

$$F(u) = \phi\{\tfrac{1}{\pi} \cosh^{-1}(-1/u)\} (1-u)^{-\frac{1}{2}}, \qquad u < 0,$$

$$F(u) = \psi\{\tfrac{1}{\pi} \cosh^{-1}(1/u)\} (1-u^2)^{-\frac{1}{2}}, \qquad u > 0,$$

with ϕ and ψ as in (9). To include completely convex functions that are not minimal, we have only to replace $F(u)\,du$ in (11) by $d\alpha(u)$, where $\alpha(u) = \int_{-1}^{u} F(t)\, dt + c \operatorname{sgn} u$.

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EIGENFUNCTION EXPANSIONS FOR NON-SYMMETRIC PARTIAL DIFFERENTIAL OPERATORS, III.*

By FELIX E. BROWDER.[1]

In two previous papers under the same title ([7], [8]), the writer has constructed a theory of eigenfunction expansions of a generalized Plancherel-Weyl type for general classes of realizations of a partial differential operator L on an open subset G of E^n, the subnormal realizations in [7] and the decomposable realizations in [8]. These results were established without any restrictions upon G, the type of L, or on the smoothness or behaviour of the coefficients of L at the boundary of G or at infinity. Results of the same type were obtained for expansions in solutions of the equations $(L-\zeta B)u=0$ for positive differential operators B.

It is the purpose of the present paper to extend and generalize these results in two separate directions. In the first place, it is very difficult to determine in any concrete case whether a particular realization of a partial differential operator under specific boundary conditions falls within either of the two classes which we have studied, except in the relatively simple case in which L is symmetric. For this reason, we have begun the study elsewhere ([5], [6], [9]) of the simplest of the basically non-symmetric cases, elliptic operators on unbounded domains under Dirichlet boundary conditions, but the results obtained at this level of generality do not as yet permit us to apply the theory as previously constructed. On the other hand, for the regular case on a bounded domain, the completeness of the eigenfunctions of the Dirichlet problem was proved by the writer in [4] by a method which extends to the whole class of regular variational boundary-value problems. It is of interest, therefore, to obtain some results on the completeness of the eigenfunctions of L for L singular but elliptic, without any assumption on the boundedness or smoothness in the large of its coefficients. In section 1, we show that if f is a distribution with compact support in G and if L satisfies certain simple local conditions at each point of G, then the orthogonality of f to each eigenfunction

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of arbitrary order of L on G implies that $f = 0$. The equivalent and somewhat more intuitive dual form of this result is that the eigenfunctions of L are complete in $C^\infty(G)$. The proof combines a refinement of the author's results in [4] with the Runge theorem for solutions of elliptic equations due to Lax [14] and Malgrange [15].

In section 2, we are concerned with another sort of extension of the eigenfunction expansion theory, the generalization of Bochner's theorem to eigenfunction expansions. Next to the Plancherel theory, the theorem of Bochner-Herglotz, which asserts that a continuous function on R^1 is positive definite if and only if it is the Fourier-Stieltjes transform of a positive finite measure, is probably the most often-applied result of classical Fourier analysis, and it is not surprising that its extension to the domain of eigenfunction expansions has attracted interest. (We remark that the proof of the Bochner theorem generalized in a rather different way to a Banach algebra context is presented in [16].) A generalization of Bochner's theorem to eigenfunction expansions for ordinary differential operators was given by M. Krein [13] in 1946, whose results also contained as a special case Bernstein's theorem on the representation of completely monotonic functions as Laplace-Stieltjes transforms of positive measures ([17], p. 160). More recently, Berezanski ([3], [4]) has given a proof of a similar theorem for elliptic differential operators from which the Bochner and Bernstein theorems follow as special cases. If L is a linear elliptic differential operator with smooth complex-valued coefficients on an open subset G of E^n, L' its formal adjoint, $\bar{L}$ the differential operator on G whose coefficients are the complex conjugates of those of L, a function $\psi(x, y, \lambda)$ on $G \times G \times R^1$ is said to be an elementary kernel for L on G if for each fixed λ in R^1, ψ is positive definite on $G \times G$ and of class C^m in x and y separately while $L_x\psi = \bar{L}_y\psi = \lambda\psi$. In terms of the notion of elementary kernel, Berezanski's theorem asserts that if $K(x, y)$ is a continuous positive-definite function on $G \times G$, then $K(x, y)$ has a representation of the form

$$K(x, y) = \int_{R^1} \psi(x, y, \lambda)\, dm(\lambda) \tag{1}$$

for some elementary kernel ψ and a positive finite measure m on R^1 if and only if

$$\int_{G\times G} K(x, y)\bar{f}(x)\,(L'g)\,(y)\,dx\,dy = \int_{G\times G} K(x, y)\,(\overline{L'f})\,(x)\,g(y)\,dx\,dy, \tag{2}$$

for every pair of functions f and g in $C_c^\infty(G)$, the family of infinitely differ-

entiable functions with compact support in G. In addition, he shows in [3] that $\psi(x,y,\lambda)$ may be written in the form $\sum_j \phi_j(x,\lambda)\bar{\phi}_j(y,\lambda)$, where $(L-\lambda I)\phi_j = 0$. In section 2 below, using a different procedure from [2] and [3], we shall establish a similar theorem for the case when L is not necessarily elliptic. The eigenfunctions which we obtain (and the elementary kernel, as well) are not, except in the hypoelliptic case, differentiable functions, but in contrast with the results for the Plancherel theory, they are locally integrable functions which are weak eigenfunctions and not distributions. We also obtain a similar representation theorem under more general hypotheses in terms of eigenfunctions with complex eigenvalues.

1. Throughout this and the following section, we shall employ the notation of [8].

Let $L = \sum_{|\alpha| \leqq 2m} a_\alpha(x) D^\alpha$ be a partial differential operator of order $2m$ $(m \geqq 1)$, on an open subset G of the Euclidean n-space E^n. We shall suppose for the sake of simplicity that the coefficients of L are infinite differentiable at each point of G, although the arguments which we put forward may be carried through under rather mild local regularity assumptions, as the perceptive reader will observe. We recall that L is elliptic at x if the $2m$-form $h(x,\xi) = \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha > 0$ for all real n-vectors $\xi \neq 0$. $(\xi^\alpha = (\xi_1)^{\alpha_1} \cdots (\xi_n)^{\alpha_n}.)$

In the present section, our basic assumptions on L are ellipticity at each point of G and uniqueness of the solution of the Cauchy problem in the small for $(L'-\zeta)$, where ζ is any complex number and L' is the adjoint differential operator to L, defined by $L'u = \sum_{|\alpha| \leqq 2m} (-1)^{|\alpha|} D^\alpha(\bar{a}_\alpha(x)u)$. To be precise, the second assumption may be formulated as follows: (U. C. P. S.): If G_0 is a connected open subset of G, ζ a complex number, and u a function in $C^\infty(G_0)$ such that $(L'-\zeta)u = 0$, and if u is identically zero on an open subset of G_0, then u is identically zero throughout G_0.

It follows from a classical argument of Holmgren (and also from the theorem that solutions of linear elliptic equations with analytic coefficients are analytic) that if L has coefficients which are real analytic functions in G, then (U. C. P. S.) holds. A more general result in the same direction follows from the recent work of Calderon [10], who has shown that if the characteristic form $h(x,\xi)$ of L' has simple characteristic roots in the complex domain with respect to every direction, then (U. C. P. S.) holds. In particular, this assumption is valid for all second-order equations. (For the bibliography of the earlier work of Muller, Heinz, Hartman and Wintner, and Aronszajn on

the second-order case, we refer to [10].) In the case of two independent variables, the result was obtained for all elliptic equations with similar restrictions on characteristic roots by Carleman. The theorems due to Carleman and Calderon are valid under very mild differentiability assumptions on the coefficients, and therefore a fortiori under our present assumption.

Under the assumption (U. C. P. S.) (restricted to $\zeta = 0$), Lax [14] and Malgrange [15] have established the following generalization of Runge's theorem for solutions of elliptic equations: Suppose the uniqueness of the solution of the Cauchy problem in the small for the elliptic operator L' and let G_0 be an open subset of G such that $G - G_0$ has no compact components in G. Then if u is a solution of $Lu = 0$ in G_0 and K is a precompact open subset of G_0, u may be approximated in $C^\infty(K)$ by a solution v of $Lv = 0$ in G. (Theorem 6, p. 341 of [15].)

The eigenfunctions of our partial differential operator L on G are defined as the solutions for some $s > 0$ and some complex ζ of the equation $(L - \zeta)^s u = 0$. Since we are assuming that the coefficients of L are infinitely differentiable and that L, and hence $(L - \zeta)^s$, is elliptic, we may restrict our attention to eigenfunctions lying in $C^\infty(G)$. The order of an eigenfunction u is the least positive s for which there exists a complex ζ with $(L - \zeta I)^s u = 0$. From the Lax-Malgrange generalization of the Runge theorem, we deduce the following result which will be useful in the proof of the main theorem of this section:

LEMMA 1. *Let L be a linear elliptic differential operator on G with infinitely differentiable coefficients for which (U. C. P. S.) holds, G_0 an open subset of G such that $G - G_0$ has no compact components in G, K a precompact open subset of G_0. If u is an eigenfunction of order s of L with eigenvalue ζ on G_0, then u may be approximated with arbitrary closeness in $C^\infty(K)$ by eigenfunctions v of L of order $\leqq s$ and eigenvalue ζ on the larger domain G.*

Proof. Since $(L - \zeta)^s$ is an elliptic differential operator, it suffices by the Runge theorem to show that if w is a solution of the equation $((L - \zeta)^s)' w = 0$ in a connected open subset G_1 of G, and if w vanishes on an open subset of G_1, then w vanishes identically on G_1. We remark first that $((L - \zeta)^s)' = (L' - \overline{\zeta})^s$. Let $w_1 = (L' - \overline{\zeta})^{s-1} w$. Then w_1 is a solution of $(L - \overline{\zeta}) w_1 = 0$ in G_1 which vanishes on an open subset of the connected open set G_1. By the assumption (U. C. P. S.), w_1 must vanish everywhere on G_1, i.e., $(L' - \overline{\zeta})^{s-1} w = 0$ in G_1. Proceeding by induction, it follows that $w = 0$, and the proof is complete.

LEMMA 2. *Let G be an open subset of E^n, K a precompact subset of G. Then there exists a pre-compact open subset G_0 of G which contains K for which $G - G_0$ has no compact components.*

Proof of Lemma 2. We may assume K compact. Let $\epsilon > 0$ be less than the positive distance from the compact set K to $E^n - G$. For each x in K, let $B_\epsilon(x) = \{y: d(x,y) < \epsilon\}$. By compactness, K may be covered by a finite family $\{B_1, \cdots, B_k\}$ of these ϵ-disks, each of which is contained in G, together with its closure. Let $G_0' = \bigcup_k B_k$. G_0' is a pre-compact open subset of G. Let $\{C_j\}$ be the family of compact components of $G - G_0'$. Each C_j is compact and open in $G - G_0'$. (To verify the last fact, we need only show that $G - G_0'$ is locally connected. But $G - \mathrm{cl}(G_0')$ being an open subset of E^n is locally connected, and a sufficiently small neighborhood in $G - G_0'$ of a point on the boundary of G_0' is identical with a neighborhood of that point in $E^n - G_0'$, and the latter is locally connected.) Since $G - G_0'$ is open in $E^n - G_0'$, each C_j is open in $E^n - G_0'$, and hence, being compact, connected, and open in $E^n - G_0'$, must be a component of $E^n - G_0'$. But the latter set has only a finite number of components, so that there are only finitely many C_j and the distance from any C_j to the non-compact components of $G - G_0'$ has a positive lower bound ϵ_1. If we set $G_0 = G_0' \cup \bigcup_j C_j$, then $G - G_0$ has no compact components and, since G_0 is bounded and its ϵ_2-neighborhood is contained in G for $\epsilon_2 = \min(\epsilon, \epsilon_1)$, G_0 is a pre-compact subset of G which contains K.

If G_0 is a pre-compact subset of G, the coefficients of our differential operator L have all their derivatives continuous and uniformly bounded on G_0. If we consider the Dirichlet problem in the variational sense on G_0, as in [4], for the elliptic operator L, it will be regular in the sense that if (ϕ, ψ) is the inner product in $L^2(G_0)$, there exist $c_0 > 0$, $k \geqq 0$ such that for all ϕ in $C_c^\infty(G_0)$, $(-1)^m \mathrm{Re}(L\phi, \phi) + k(\phi, \phi) \geqq c_0 \sum_{|\beta| \leqq m} (D^\beta \phi, D^\beta \phi)$. It follows that the generalized Friedrichs extension of the operator $(L + (-1)^m kI)$ with domain $C_c^\infty(G_0)$ has a compact inverse R defined on the whole of $L^2(G_0)$.

Let $A_{2m} = \sum_{|\beta| \leqq m} (-1)^{|\beta|} D^\beta D^\beta$. Using the Fourier transform of tempered distributions, one may construct an elementary solution $e_{2m}(x - y)$ for A_{2m} on $L^2(E^n)$ which lies in C^∞ off the diagonal and such that, if B is any disk containing the closure of G_0 in its interior and $\epsilon > 0$, then for x and y in B, $|e_{2m}(x - y)| \leqq K_\epsilon\{|x - y|^{2m-n-\epsilon} + 1\}$. It follows by a standard argument that there exists a C^∞ function $r(x, y)$ on $G \times G$, such that if u is a distribution solution of the equation $A_{2m} u = f$ for an integrable function on B, then

$u(x) = \int_B e_{2m}(x-y)f(y)\,dy + \int_B r(x,y)u(y)\,dy$, for almost all x in G_0. If R_2 is a bounded linear mapping of $L^2(G_0)$ into itself such that $A_m R f = f$, then on G_0, $(Rf)(x) = E(f)(x)$, where E is the integral operator with kernel $e_{2m}(x-y) + (R_2^*)_y r(x,y)$. If P is the projection mapping of $L^2(B)$ on $L^2(G_0)$ obtained by restriction, it follows that for $2mj > n/2$, $(PR_2P)^j = (PEP)^j$ is an operator of Hilbert-Schmidt type.

THEOREM 1. *The eigenfunctions of the Dirichlet problem for L (i.e., for the generalized Friedrichs extension of L restricted to $C_c^\infty(G_0)$) on the relatively compact open subset G_0 of G are complete in $L^2(G_0)$ and in $C^\infty(G_0)$ in their respective topologies.*

Proof of Theorem 1. Completeness in $L^2(G_0)$ will follow from the proof of Theorem 1 of [4] and the following lemma:

LEMMA 3. *For sufficiently large j, more specifically $2mj > n/2$, R^j is an operator of Hilbert-Schmidt type.*

Proof of Lemma 3. It follows from the argument of [4] that $R = SR_1$, where S is a bounded operator in $L^2(G_0)$ and R_1 in the inverse of $(L_s + (-1)^m kI)$, with L_s the Friedrichs extension of the symmetric part of L, taken as a semi-bounded operator with domain $C_c^\infty(G_0)$. It suffices to show that R_1^j is an operator of Hilbert-Schmidt class for j large. For then, by the minimax principle for the eigenvalues of a compact positive self-adjoint operator, the n-th eigenvalue (with repetitions for multiplicity) of $R^*R = R_1^* \cdot S^* \cdot S \cdot R_1$ is less than $\| S \|^2$ times the n-th eigenvalue in decreasing order of R_1^2. Since R_1^j is of Hilbert-Schmidt type, $\mathrm{Tr}(R_1^{2j}) < \infty$, where $\mathrm{Tr}(T)$ is the trace of the compact positive operator T. It follows that $\mathrm{Tr}((R^*R)^j) < \infty$. By a theorem of Ky Fan [12], it follows that $\mathrm{Tr}((R^*)^j R^j) \leqq \mathrm{Tr}((R^*R)^j) < \infty$. Thus $(R^*)^j$ and hence also its adjoint R^j are operators of Hilbert-Schmidt type.

Let B be an open disk in E^n containing the closure of G_0 in its interior. If $H_m(G_0)$ and $H_m(B)$ are the spaces of functions on G_0 and B respectively satisfying the m-th order null Dirichlet conditions in the variational sense, we may describe these spaces as the closures of $C_c^\infty(G_0)$ and $C_c^\infty(B)$ in the norm $\| f \|_m^2 = \sum_{|\beta| \leqq m} (D^\beta f, D^\beta f)$ associated with the inner product $[f,g]_m = \sum_{|\beta| \leqq m} (D^\beta f, D^\beta g)$. In particular, $H_m(G_0)$ may be taken as a closed subspace of $H_m(B)$. If we define the Dirichlet form associated with $(L_s + (-1^m kI)$ as $d(f,g) = \int_{G_0} \{(-1)^m L_s f + kf\}\bar{g}\,dx$, it follows by integration by parts and the

Schwarz inequality that for a suitable positive constant c_0, $d(f,f) \geqq c_0 \| f \|_m^2$. Let R_2 be the inverse of the Friedrichs extension of the differential operator $\sum_{|\beta| \leqq m} (-1)^{|\beta|} D^\beta (D^\beta u)$. R_1 and R_2 are characterized, respectively, by the equations:

$$d(R_1 f, \phi) = (-1)^m (f, \phi),\ \phi \text{ in } H_m(G_0);\ R_1 f \in H_m(G_0);$$

$$[R_2 f, \phi]_m = (f, \phi),\ \phi \in H_m(B);\ R_2 f \in H_m(B).$$

It follows that for f in $H(G_0)$,

$$\begin{aligned}(-1)^m (R_1 f, f) &= d(R_1 f, R_1 f) = \sup\{|\, d(R_1 f, \phi)|^2 \cdot d(\phi, \phi)^{-1} : \phi \in C_c^\infty(G_0)\} \\ &= \sup\{|\, (f, \phi)|^2 \cdot d(\phi, \phi)^{-1} : \phi \in C_c^\infty(G_0)\} \\ &\leqq \sup\{|\, (f, \phi)|^2 \cdot c_0^{-1} \| \phi \|_m^{-2} : \phi \in C_c^\infty(G_0)\} \\ &= c_0^{-1} \sup\{|\, [R_2 f, \phi]_m|^2 \cdot \| \phi \|_m^{-2} : \phi \in C_c^\infty(G_0)\} \\ &\leqq c_0^{-1} [R_2 f, R_2 f]_m = c_0^{-1} (R_2 f, f).\end{aligned}$$

This last chain of inequalities yields the conclusion that in the Hilbert space $L^2(G_0)$, we have $0 \leqq (-1)^m R_1 \leqq c_0^{-1}(P R_2 P)$. But by the remarks preceeding Theorem 1, for $2mj > n/2$, $(PR_2P)^j$ is an operator of Hilbert-Schmidt type. Hence so is $R_1{}^j$, and the lemma is proved.

Proof of Theorem 1 *continued.* To complete the proof, we must show that the eigenfunctions of L under null Dirichlet conditions of order m are dense in $C^\infty(G_0)$. It suffices to show, for any ψ in $C_c^\infty(G)$, a non-negative integer j_0 and a compact subset K of G_0, that ψ can be approximated by a linear combination of eigenfunctions of L in $C^{j_0}(K)$. Let j be an integer such that $2mj > n/2 + j_0$. It is a standard result of the theory of elliptic equations (see for example the writer's paper in the Annals of Mathematics Study, No. 33, Princeton, 1954) that R^j for such j is a bounded linear mapping of $L^2(G_0)$ into $C^{j_0}(K)$. Let $\phi = (L + (-1)^m k I)^j \psi$. Since ϕ lies in $L^2(G_0)$, there exists a finite family of eigenfunctions $\{u_r\}$ of the Dirichlet problem for L on G_0 such that $\| \phi - \sum_r u_r \|_{L^2(G_0)} < \epsilon$. For each eigenvalue ζ of L, $\zeta \neq (-1)^{m+1} k$. If u is an eigenfunction of order $\leqq s$ with eigenvalue ζ, then $u = (L + (-1)^m k I)^j v$, where v is a finite sum of eigenfunctions with eigenvalue ζ of order $\leqq s$. Indeed let V_s be the finite subspace of $L^2(G_0)$ spanned by the eigenfunctions of L with eigenvalue ζ of order $\leqq s$. Then L maps V_s into itself, and $(L - \zeta I)^s (V_s) = \{0\}$. Thus ζ is the only point in the spectrum of L on V_s, and $(L + (-1)^m k I)^{-1}$ is well-defined and maps V_s onto itself. Iterating j times, we obtain our previous observation. It follows

that for each u_r, there exists a finite sum v_r of eigenfunctions of L with the same eigenvalue such that $(L+(-1)^m kI)v_r = u_r$. But then

$$\psi - \sum_r v_r = R^j(\phi - \sum_r u_r).$$

By our previous remark,

$$\sup_{x \in K} |D^\alpha(\psi - \sum_r v_r)(x)| \leqq M_{\alpha,K} \| \phi - \sum_r u_r \|_{L^2(G_0)} \leqq M_{\alpha,K}\epsilon.$$

If we now choose ϵ sufficiently small and a corersponding set of eigenfunctions $\{u_r\}$, we may make any finite number of the bounds $M_{\alpha,K}\epsilon$ arbitrarily small.

THEOREM 2. *Let L be an elliptic differential operator with infinitely differentiable coefficients in the interior of an open set G of E^n and satisfying the (U.C.P.S.) property above. Then the eigenfunctions of L on G are dense in $C^\infty(G)$. Dually, if w is a distribution with compact support in G such that $w(u) = 0$ for all eigenfunctions u of L on G, then $w = 0$.*

Proof of Theorem 2. Since $C_c^\infty(G)$ is dense in $C^\infty(G)$, it suffices to approximate ψ in $C_c^\infty(G)$, uniformly with all derivatives up to some given order on a compact set K. By Lemma 2, there exists a pre-compact open subset G_0 of G containing K and such that $G - G_0$ has no compact components. Since K is a compact subset of G_0, by Theorem 1 we can approximate ψ in $C_c^\infty(G_0)$ by a finite sum of eigenfunctions u_r of L on G_0. But by Lemma 1, each of these eigenfunctions can be approximated with arbitrary accuracy in $C^\infty(K)$ by an eigenfunction u_r' of L on G, since $G - G_0$ has no compact components. Combining these approximations, the theorem is proved.

2. The present section is concerned, as we have remarked in the Introduction, with the proof of an extension of Bochner's theorem to eigenfunction expanstions for a partial differential operator L, not necessarily elliptic, on an open subset G of E^n.

Let L be of order r $(r \geqq 1)$, $L = \sum_{|\alpha| \leqq r} a_\alpha(x) D^\alpha$, We shall assume once more that the coefficients of L are infinitely differentiable at each point of G. The adjoint operator L' is defined by $L'u = \sum_{|\alpha| \leqq r} (-1)^{|\alpha|} D^\alpha(\bar{a}_\alpha(x)u)$, and $\bar{L}$ is the operator obtained from L by taking the complex conjugates of the coefficients.

Let $K(x, y)$ be a continuous function on $G \times G$. The function K is said to be positive definite if for every finite set $\{x_1, \cdots, x_s\}$ of points of G and a corresponding set of complex numbers $\{\zeta_1, \cdots, \zeta_s\}$, we have

$$\sum_{j,k=1}^{s} K(x_j, x_k)\zeta_j\bar{\zeta}_k \geqq 0. \tag{3}$$

If K is positive definite, the inner product defined by

$$\langle f, g\rangle = \int_{G\times G} K(x,y)f(x)\bar{g}(y)\,dx\,dy \tag{4}$$

for f and g in $C_c^\infty(G_0)$, defines a semi-norm $\| f \|_K = \langle f, f\rangle^{\frac{1}{2}}$ on $C_c^\infty(G_0)$, since by (3), $\langle f, f\rangle \geqq 0$, and by standard arguments, $|\langle f, g\rangle| \leqq \| f \|_K \cdot \| g \|_K$ and $\| f+g \|_K \leqq \| f \|_K + \| g \|_K$.

Let N be the linear subset of $C_c^\infty(G)$ of those f for which $\| f \|_K = 0$. Since $\langle f, h\rangle = 0$ for f in N and h in $C_c^\infty(G)$, the inner product $\langle -, -\rangle$ is defined on the equivalence classes of $C_c^\infty(G)$ modulo N and yields a pre-Hilbert space structure on $C_c^\infty(G)/N$. Let the Hilbert space H be the completion of $C_c^\infty(G)/N$ with respect to $\| - \|_K$.

Suppose that we are given a linear endomorphism T_0 of $C_c^\infty(G)$ and another linear endomorphism $T_0^\#$ of $C_c^\infty(G)$ which is adjoint to T_0 with respect to the K-inner product, i.e.,

$$\langle T_0 f, g\rangle = \langle f, T_0^\# g\rangle \tag{5}$$

for all f and g in $C_c^\infty(G)$. Then for $f \in N$, (5) implies that $\langle T_0 f, g\rangle = 0$ for all g in $C_c^\infty(G)$, i.e., $T_0 f \in N$. Thus $T_0(N) \subset N$, $T_0^\#(N) \subset N$. As a result T_0 and $T_0^\#$ induce linear endomorphisms of $C_c^\infty(G)/N$. We denote these endomorphisms by T and $T^\#$ and consider them as linear transformations defined on the dense subset $C_c^\infty(G)/N$ of H and with range in H. If we denote the inner product in H by $\langle \cdot, \cdot \rangle$, then (5) implies that

$$\langle Tu, v\rangle = \langle u, T^\# v\rangle;\ u, v \in D(T) = D(T^\#). \tag{6}$$

Definition 1. T_0 is said to be symmetric with respect to K if we may choose $T_0^\# = T_0$.

In particular, the partial differential operator $\tilde{L}'$ is symmetric with respect to K if and only if it satisfies the condition (2) given in the Introduction.

Definition 2. T_0 is said to be subnormal if there exists a Hilbert space H_1 containing H as a closed subspace and a normal operator T_1 in H_1 such that $T \subseteq T_1$.

Definition 3. If u is a locally integrable function on G, then u is said to be an eigenfunction of T_0', the dual of T_0, with eigenvalue ζ, a complex number, if

$$(7) \qquad \int_G u(x)\,(T_0 f)\,(x)\,dx = \zeta \int_G u(x) f(x)\,dx$$

for all f in $C_c^\infty(G)$.

Remark. In the present section, we consider only eigenfunctions of order one in the sense of section 1 of the present paper, or [8]. Further, while we allow eigenfunctions in the distribution sense, we restrict ourselves to locally integrable functions instead of allowing general distributions.

Definition 4. If $\psi(x,y,\lambda)$ is a function on $G \times G \times R^1$ integrable on compact subsets of $G \times G$ for each λ in R^1, then ψ is said to be an elementary kernel for T_0 on G if for each pair f and g in $C_c^\infty(G)$,

$$(8)_1 \qquad \int_{G\times G} \psi(x,y,\lambda)\,(T_0 f)\,(x)\,\bar{g}(y)\,dx\,dy$$
$$= \int_{G\times G} \psi(x,y,\lambda) f(x)\,(\overline{T_0 g})\,(y)\,dx\,dy$$
$$= \lambda \int_{G\times G} \psi(x,y,\lambda) f(x)\,\bar{g}(y)\,dx\,dy.$$

$$(8)_2 \qquad \int_{G\times G} \psi(x,y,\lambda) f(x)\,\bar{f}(y)\,dx\,dy \geqq 0.$$

for all λ in R^1.

(To obtain a corresponding definition for complex eigenvalues ζ, we replace equation $(8)_1$ by two equations,

$$(8)_3 \qquad \int_{G\times G} \psi(x,y,\zeta)\,(T_0 f)\,(x)\,\bar{g}(y)\,dx\,dy$$
$$= \zeta \int_{G\times G} \psi(x,y,\zeta) f(x)\,\bar{g}(y)\,dx\,dy.$$

and

$$(8)_4 \qquad \int_{G\times G} \psi(x,y,\zeta) f(x)\,(\overline{T_0 g})\,dx\,dy$$
$$= \bar{\zeta} \int_{G\times G} \psi(x,y,\zeta) f(x)\,\bar{g}(y)\,dx\,dy.)$$

THEOREM 3. *Let $K(x,y)$ be a continuous function on $G \times G$, T_0 a continuous linear endomorphism of $C_c^\infty(G)$. Then K is positive definite on G and T_0 is symmetric with respect to K if and only if there exist a positive finite Borel measure m on R^1, an elementary kernel $\psi(x,y,\lambda)$ for T_0 on G, and a (possibly infinite) sequence $\{\phi_j(x,\lambda)\}$ of eigenfunctions of the dual of T_0 on G with eigenvalue λ for each λ in R^1 such that:*

(a) *$\psi(x,y,\lambda)$ is measurable on $G_B \times G \times R^1$ with respect to the measure*

$(dx \cdot dy \cdot m(d\lambda))$. Each ϕ_j is measurable on $G \times R^1$ with respect to $dx \cdot m(d\lambda)$. For each j, $\int_{R^1} |\phi_j(x,\lambda)|^2 m(d\lambda)$ is bounded a.e. on every compact subset C of G, and we have

$$\sum_j \text{ess. sup.}_{x \in C} \int_{R^1} |\phi_j(x,\lambda)|^2 m(d\lambda) < \infty. \tag{9}$$

In addition,

$$\text{ess. sup.}_{x,y \in C} \int_{R^1} |\psi(x,y,\lambda)| m(d\lambda) < \infty. \tag{10}$$

(b) *There exists the representation*

$$\psi(x,y,\lambda) = \sum_j \phi_j(x,\lambda)\bar{\phi}_j(y,\lambda),$$

the sum being absolutely convergent outside a set of $dx \cdot dy \cdot m(d\lambda)$ measure null on $G \times G \times R^1$, and convergent in $L^1(m)$ uniformly for x and y in any compact subset of G.

(c) *For almost all x and y in G,*

$$K(x,y) = \int_{R^1} \psi(x,y,\lambda) m(d\lambda) = \sum_j \int_{R^1} \phi_j(x,\lambda)\bar{\phi}_j(y,\lambda) m(d\lambda),$$

each integral and the sum being uniformly absolutely convergent a.e. on every compact subset of $G \times G$.

(d) *If C is a compact subset of G, d_0 its distance from $E^n - G$, h an element of E^n with $|h| < d_0$, then*

$$\sum_j \text{ess. sup.}_{x \in C} \int_{R^1} |\phi_j(x+h) - \phi_j(x)|^2 m(d\lambda) = r_C(h) < \infty,$$

where $r_C(h) \to 0$ as $h \to 0$.

THEOREM 4. *Let L be a partial differential operator with infinitely differentiable coefficients in G and with the property that for each λ in R^1, every distribution solution of $(L - \lambda I)\phi = 0$ is an infinitely differentiable function. If $T_0 = \bar{L}'$ satisfies the conditions of Theorem 1, i.e., is symmetric with respect to a continuous positive definite function K on $G \times G$, then the elementary kernel $\psi(x,y,\lambda)$ may be chosen infinitely differentiable in $G \times G$ for each λ, each of the eigenfunctions $\phi_j(x,\lambda)$ may be chosen infinitely differentiable in G for each λ in R^1, $L_x\psi = \bar{L}_y\psi = \lambda\psi$, $L\phi_j = \lambda\phi_j$ for all λ. In this case,*

$$\psi(x,y,\lambda) = \sum_j \phi_j(x,\lambda)\bar{\phi}_j(y,\lambda), \tag{11}$$

the sum converging uniformly for x and y in any compact subset of G for each λ in R^1. For fixed λ in R^1 and y in G, the right-hand side of (11) *converges to ψ in $C^\infty(G)$ as a function of x in G. Similarly, for fixed λ in R^1 and x in G, the right-hand side of* (11) *converges to ψ in $C^\infty(G)$ as a function of y in G.*

THEOREM 5. *If K is a continuous positive definite function on G and T_0 is subnormal with respect to K, then one may obtain a representation of K by a positive measure m on C^1, a complex elementary kernel $\psi(x,y,\zeta)$ for T_0 on G, and a sequence of eigenfunctions $\{\phi_j(x,\zeta)\}$ of the dual of T_0 with complex eigenvalues, for which conclusions* (a)-(d) *of Theorem* 1 *are satisfied with all integrations in $m(d\zeta)$ being taken over the complex plane C^1.*

Proof of Theorem 3. The fact that conditions (a), (b), (c), and (d) taken together imply that K is continuous and positive definite and that T_0 is symmetric with respect to K, follows by an elementary computation. We shall, therefore, restrict ourselves to the proof that, if K is continuous in $G \times G$ and positive definite, and if T_0 is symmetric with respect to K, then conditions (a), (b), (c), and (d) must hold.

If T_0 is symmetric with respect to K, the corresponding operator T in the Hilbert space H, defined earlier in this section, is a densely defined symmetric operator in H. By a theorem of Neumark ([1]), there exists a Hilbert space H_1 containing H as a closed subspace and a self-adjoint operator T_1 in H_1 whose restriction to $D(T_1) \cap H$ contains the operator T. Let $\{E_\lambda\}$ be the spectral family of T_1 in H_1. If u is any element of H_1, the cyclic subspace generated by u with respect to $\{E_\lambda\}$ is defined as the subspace of H_1 spanned by the elements $E(S)u$, where S runs over the Borel subsets of R^1. It follows by a simple argument (see Lemma 3 of [7]) that it is possible to choose H_1 and T_1 such that H_1 is the direct sum of a countable family of orthogonal cyclic subspaces H_j corresponding to a sequence of elements $\{u_j\}$ of H_1. Let $m_j(S) = \langle E(S)u_j, u_j\rangle$, and set

$$m(S) = \sum_j 2^{-j} m_j(S) \tag{12}$$

for any Borel set S of R^1.

By a standard argument of the spectral theory of self-adjoint operators, there exists an unitary mapping U_j of H_j onto $L^2(m_j)$, where for a bounded function c on R^1, $U_j^{-1}(c) = \int_{R^1} c(\lambda)\, dE_\lambda u_j$. Further, if P_j is the orthogonal

projection of H_1 on its closed subspace H_j, $P_j(D(T_1)) = D(T_1) \cap H_j$. For u in $D(T_1)$, $P_j T_1 u = T_1 P_j u$, while $(U_j P_j T_1 u)(\lambda) = \lambda (U_j P_j u)(\lambda)$ a.e. in m_j for each u in $D(T_1)$.

By the definition (12) of the measure m, m_j is absolutely continuous with respect to m. If h_j is the Radon-Nikodym derivative of m_j with respect to m, h_j is defined a.e. in m and $h_j \geqq 0$. For c in $L^2(m_j)$, put $S_j c = h_j^{\frac{1}{2}} c$. Then, S_j is an isometric mapping of $L^2(m_j)$ into $L^2(m)$, and $(S_j U_j P_j T_1 u)(\lambda) = \lambda (S_j U_j P_j u)(\lambda)$ a.e. in m for each u in $D(T_1)$.

Let R be the mapping of $C_c^\infty(G)$ into H which sends a function f into its equivalence class in $C_c^\infty(G)/N$, where the latter is considered as a subset of H. If C is a fixed open subset precompact in G, then for $f \in C_c^\infty(C)$,

$$\| Rf \|_K^2 = \int_{G \times G} K(x,y) f(x) \bar{f}(y)\, dx\, dy \leqq a_C^2 \{ \| f \|_{L^1(C)} \}^2,$$

where a_C^2 is the upper bound of $| K(x,y) |$ on $C \times C$.

We complete our notational definitions by setting

$$Q_j f = S_j U_j P_j R f, \qquad f \in C_c^\infty(G). \tag{13}$$

Q_j maps $C_c^\infty(G)$ into $L^2(m)$.

Since $H_1 = \sum_j \oplus H_j$, U_j is unitary, and S_j is isometric, we have

$$\sum_j \| Q_j f \|_{L^2(m)}^2 = \| Rf \|_K^2 \leqq a_C^2 \{ \| f \|_{L^1(C)} \}^2, \qquad f \in C_c^\infty(C). \tag{14}$$

By continuity, Q_j may be extended from the dense subset $C_c^\infty(C)$ of $L^1(C)$ to a bounded linear transformation from $L^1(C)$ to $L^2(m)$ with the same bound. By a theorem of Dunford and Pettis [11], every bounded linear transformation on Q_j from $L^1(C)$ into $L^2(m)$ has an integral representation of the form

$$(Q_j f)(\lambda) = \int_G \phi_j(x,\lambda) f(x)\, dx, \qquad f \in L^1(C), \tag{15}$$

where $\phi_j(x,\lambda)$ is a measurable function on $C \times R^1$ with respect to the measure $dx \cdot m(d\lambda)$, and a.e. in G,

$$\int_{R^1} | \phi_j(x,\lambda) |^2\, m(d\lambda) \leqq b_{C,j}^2, \tag{16}$$

where $b_{C,j}$ is the norm of Q_j as a mapping from $L^1(C)$ into $L^2(m)$. By (14), $\sum_j b_{C,j}^2 \leqq a_C^2$, i.e.,

$$\sum_j \operatorname{ess.sup}_{x \in C} \int_{R^1} | \phi_j(x,\lambda) |^2\, m(d\lambda) \leqq a_C^2. \tag{17}$$

We now remark that if $C \subset C_1$, where C_1 is another precompact open subset of G, then the $\phi_j'(x,\lambda)$ defined for C_1 by the same procedure coincide on $C \times R^1$ with $\phi_j(x,\lambda)$ except on a set of $dx \cdot m(d\lambda)$ measure zero. Since G is the union of an increasing sequence of pre-compact subsets, we may define $\phi_j(x,\lambda)$ on $G \times R^1$ uniquely up to a set of $dx \cdot m(d\lambda)$ measure zero and satisfying (14) for every f in $C_c^\infty(G)$, and all the other conditions on every compact subset C of G.

If $f \in C_c^\infty(G)$,

$$(Q_j T_0 f)(\lambda) = (S_j U_j P_j R T_0 f)(\lambda) = (S_j U_j P_j T R f)(\lambda)$$
$$= (S_j U_j T_1 P_j R f)(\lambda) = \lambda (S_j U_j P_j R f)(\lambda) = \lambda (Q_j f)(\lambda)$$

a.e. in m. Thus for each f in $C_c^\infty(G)$, there exists an m-null set $\pi(f)$ such that for $\lambda \in R^1 - \pi(f)$,

$$\int_G \phi_j(x,\lambda)(T_0 f)(x)\,dx = \lambda \int_G \phi_j(x,\lambda) f(x)\,dx. \tag{18}$$

By an elementary proof given in [8], thre exists a dense denumerable set $\{f_k\}$ in $C_c^\infty(G)$. Let $\pi_k = \pi(f_k)$, $\pi = \bigcup_k \pi_k$. Then for $\lambda \in R^1 - \pi$, it follows from the continuity of T_0 as an endomorphism of $C_c^\infty(G)$, that (18) holds for all f in $C_c^\infty(G)$. If we set $\phi_j(x,\lambda) = 0$ for $\lambda \in \pi$ and all x in G, the change in ϕ_j affects its value only on an $(dx \cdot m(d\lambda))$-null set and leaves the measurability properties and the previously discussed properties of the ϕ_j unaltered. After the change, (18) holds for all f in $C_c^\infty(G)$ and all λ in R^1, i.e., $\phi_j(x,\lambda)$ is an eigenfunction of the dual of T_0 with eigenvalue λ in the sense of Definition 3 for every λ.

Let $q(x,\lambda) = \sum_j |\phi_j(x,\lambda)|^2$, $q_N(x,\lambda) = \sum_j |\phi_j(x,\lambda)|^2$. By (17)

$$\text{ess. sup.}_{x \in C} \int_{R^1} q_N(x,\lambda)\, m(d\lambda) \leqq a_C^2. \tag{19}$$

Since q_N is a monotone non-decreasing sequence of nonnegative functions converging everywhere to q as $N \to \infty$, it follows that q is a measurable function on $G \times R^1$ with respect to $dx \cdot m(d\lambda)$, finite a.e. with respect to that measure, for which

$$\text{ess. sup.}_{x \in C} \int_{R^1} q(x,\lambda)\, m(d\lambda) \leqq a_C^2 \tag{20}$$

for every precompact open subset C of G.

Let $\psi_N(x,y,\lambda) = \sum_{j \leqq N} \phi_j(x,\lambda)\bar{\phi}_j(y,\lambda)$. By the Schwarz inequality

$$|\psi_N(x,y,\lambda)-\psi_M(x,y,\lambda)| \leqq |q_N(x,\lambda)-q_M(x,\lambda)|^{\frac{1}{2}} \cdot |q_N(y,\lambda)-q_M(y,\lambda)|^{\frac{1}{2}}.$$

It follows that $\psi_N(x,y,\lambda)$ converges to a function $\psi(x,y,\lambda)$, measurable with respect to $dx \cdot dy \cdot m(d\lambda)$ on $G \times G \times R^1$ and satisfying the inequality.

$$|\psi(x,y,\lambda)| \leqq [q(x,\lambda)]^{\frac{1}{2}} \cdot [q(y,\lambda)]^{\frac{1}{2}}. \tag{21}$$

If C is a precompact subset of G, let N_j be a null set in C such that for x in $C-N_j$,

$$\int_{R^1} |\phi_j(x,\lambda)|^2 m(d\lambda) \leqq b_{C,j}^2.$$

On the complement of the null set $N = \bigcup_j N_j$, the series

$$\sum_j \int_{R^1} |\phi_j(x,\lambda)|^2 m(d\lambda)$$

is convergent, and

$$\sum_{j=M}^{N} \sup_{x \in N} \int_{R^1} |\phi_j(x,\lambda)|^2 m(d\lambda) \leqq \sum_{j=M}^{N} b_{C,j}^2,$$

with the terms on the right going to zero as M and $N \to \infty$. But,

$$\int_{R^1} |q_N(x,\lambda)-q_M(x,\lambda)|^2 m(d\lambda) = \sum_{j=M}^{N} \int_{R^1} |\phi_j(x,\lambda)|^2 m(d\lambda),$$

implies, then, that $q_N(x,\cdot)$ converges to a limit in $L^2(m)$ uniformly for x in N. This limit must obviously be $q(x,\cdot)$ since $q_N(x,\lambda)$ converges to $q(x,\lambda)$ for every x and λ. Turning to ψ_N, we see that

$$\int_{R^1} |\psi_N(x,y,\lambda)-\psi_M(x,y,\lambda)|\, m(d\lambda) \leqq \left[\int_{R^1} |q_N(x,\lambda)-q_M(x,\lambda)|^2 m(d\lambda)\right] \cdot \left[\int_{R^1} |q_N(y,\lambda)-q_M(y,\lambda)|^2 m(d\lambda)\right],$$

showing that $\psi_N(x,y,\cdot)$ converges to a limit in $L^1(m)$ uniformly for x and y in N. This limit must be $\psi(x,y,\cdot)$ a.e. on $G \times G$, since $\psi_N(x,y,\lambda)$ converges to $\psi(x,y,\lambda)$ on $G \times G \times R^1$. We have, in addition,

$$\text{ess. sup.}_{x,y \in C} \int_{R^1} |\psi(x,y,\lambda)|\, m(d\lambda) < \infty. \tag{22}$$

Let f and g be arbitrary elements of $C_c^\infty(G)$. It follows from the above inequalities that

$$\int_{G \times G \times R^1} |\psi_N(x,y,\lambda)-\psi(x,y,\lambda)| \cdot |T_0 f(x)| \cdot |g(y)|\, dx\, dy\, m(d\lambda)$$

goes to zero as $N \to \infty$. It follows that there exists a null set $\pi_{f,g}$ with respect to m on R^1 and a subsequence $\{N_k\}$ such that

$$\int_{G\times G} |\psi(x,y,\lambda) - \psi_{N_k}(x,y,\lambda)| \cdot |T_0 f(x)| \cdot |g(y)|\, dx\, dy \to 0$$

as $k \to \infty$ for $\lambda \in R^1 - \pi_{f,g}$. Similarly (possibly choosing a subsequence),

$$\int_{G\times G} |\psi(x,y,\lambda) - \psi_{N_k}(x,y,\lambda)| \cdot |f(x)| \cdot |T_0 g(y)|\, dx\, dy \to 0$$

for λ in the complement of another m-null set $\pi'_{f,g}$, which we may assume the same as the first. Finally, we may assume (again possibly enlarging the m-null set $\pi_{f,g}$) that for the subsequence N_k,

$$\int_{G\times G} |\psi(x,y,\lambda) - \psi_{N_k}(x,y,\lambda)| \cdot |f(x)| \cdot |g(y)|\, dx\, dy \to 0 \text{ as } k \to \infty.$$

Thus we have

$$\int_{G\times G} \psi(x,y,\lambda) T_0 f(x) \bar{g}(y)\, dx\, dy = \lim_k \int_{G\times G} \psi_{N_k}(x,y,\lambda) T_0 f(x) \bar{g}(y)\, dx\, dy$$

$$= \lambda \lim_k \int_{G\times G} \psi_{N_k}(x,y,\lambda) f(x) \bar{g}(y)\, dx\, dy = \lambda \int_{G\times G} \psi(x,y,\lambda) f(x) \bar{g}(y)\, dx\, dy,$$

for λ outside $\pi_{f,g}$. By a similar argument,

$$\int_{G\times G} \psi(x,y,\lambda) f(x) \overline{(T_0 g)}(y)\, dx\, dy = \lambda \int_{G\times G} \psi(x,y,\lambda) f(x) \bar{g}(y)\, dx\, dy,$$

for λ in the complement of $\pi_{f,g}$. Let f and g run independently through a dense denumerable set $\{f_k\}$ in $C_c^\infty(G)$, and let π be the union of a corresponding sequence of m-null sets $\pi_{f,g}$. In the complement of π, we have by a simple continuity argument,

$$\begin{aligned} &\int_{G\times G} \psi(x,y,\lambda) (T_0 f)(x) \bar{g}(y)\, dx\, dy \\ &\quad = \int_{G\times G} \psi(x,y,\lambda) f(x) \overline{(T_0 g)}(y)\, dx\, dy \\ &\quad = \lambda \int_{G\times G} \psi(x,y,\lambda) f(x) \bar{g}(y)\, dx\, dy, \end{aligned} \tag{23}$$

for all f and g in $C_c^\infty(G)$. If we set $\psi(x,y,\lambda) = 0$ on $G \times G \times \pi$, which is a null set with respect to $dx \cdot dy \cdot m(d\lambda)$, none of our previous conclusions are disturbed, and the equality (23) will hold after the change for all λ in R^1.

It then follows that $\psi(x, y, \lambda)$ is an elementary kernel for T_0 in the sense of Definition 4.

If we set $S(x, y) = \int_{R^1} \psi(x, y, \lambda) m(d\lambda)$, the inequality (22) implies that $S(x, y)$ is essentially bounded on every compact subset of $G \times G$. By an elementary analytical argument,

$$S(x, y) = \sum_j \int_{R^1} \phi_j(x, \lambda)\, \bar{\phi}_j(y, \lambda) m(d\lambda),$$

with each integral, as well as the sum, being uniformly absolutely convergent a.e. on each compact subset of $G \times G$. To complete the verification of condition (a) of Theorem 1, we must show that $S(x, y) = K(x, y)$ a.e. on $G \times G$. Let f and g be two elements of $C_c^\infty(G)$. By the Fubini theorem,

$$\begin{aligned}
&\int_{G\times G} S(x, y) f(x) \bar{g}(y)\, dx\, dy \\
&\qquad = \lim_N \sum_{j \leqq N} \int_{R^1} \Big(\int_G \phi_j(x, \lambda) f(x)\, dx \Big) \cdot \Big(\int_G \bar{\phi}_j(y, \lambda) \bar{g}(y)\, dy \Big) \\
&\qquad = \sum_j \int_{R^1} Q_j(f)(\lambda) Q_j(g)(\lambda) m(d\lambda) = \langle Rf, Rg \rangle.
\end{aligned}$$

But by the definition of the K-inner product,

$$\langle Rf, Rg \rangle = \int_{G\times G} K(x, y) f(x) \bar{g}(y)\, dx\, dy.$$

Since the finite linear combinations of products $f(x)\bar{g}(y)$ with f and g in $C_c^\infty(G)$ are dense in $L^2(C \times C)$ for combat subsets C of G, and since both $K(x, y)$ and $S(x, y)$ are essentially uniformly bounded on any compact subset of $G \times G$, the equality,

$$\int_{G\times G} [K(x, y) - S(x, y)] f(x) \bar{g}(y)\, dx\, dy = 0$$

for all f and g in $C_c^\infty(G)$, implies that $K(x, y) = S(x, y)$ a.e. in $G \times G$.

To complete the proof of Theorem 3, we must establish (d). Let C be a compact subset of G, d_0 its distance from $E^n - G$, h in E^n with $|h| = d < d_0$. The translation mapping $T_h f(x) = f(x - h)$ is a continuous linear mapping of $C_c^\infty(C)$ into $C_c^\infty(G)$. For f in $C_c^\infty(C)$, we have

$$\| RT_h f - Rf \|_K^2 = \int_{G\times G} K(x,y)[f(x-h)-f(x)][\bar{f}(y-h)-\bar{f}(y)]\,dx\,dy$$

$$= \int_{G\times G} [K(x+h,y+h)+K(x,y)-K(x+h,y) - K(x,y+b)]f(x)\bar{f}(y)\,dx\,dy$$

$$\leqq \sup_{x,y\in N_d(C)} |K(x+h,y+h)+K(x,y) - K(x+h,y)-K(x,y+h)| \cdot \| f \|_{L^1(C)}^2,$$

where $N_d(C)$ is the set of points at distance at most d from C.

If we set $r_C(h)$ equal to the supremum term in the last inequality, we have

$$\| Q_j T_h f - Q_j f \|_{L^2(m)}^2 \leqq r_C(h) \| f \|_{L^1(C)}^2.$$

By our previous remarks, however,

$$(Q_j T_h f - Q_j f)(\lambda) = \int_G \phi_j(x,\lambda)\cdot[f(x-h)-f(x)]\,dx$$

$$= \int_G [q(x+h,\lambda)-q(x,\lambda)]f(x)\,dx.$$

It follows from the Dunford-Pettis theorem that

$$\sum_j \text{ess. sup.}_{x\in C} \int_{R^1} |\phi_j(x+h)-\phi_j(x)|^2\, m(d\lambda) \leqq r_C(h),$$

while the fact that $r_C(h)\to 0$ follows from the continuity of K and the precompactness of $N_d(C)$.

Thus the proof of Theorem 3 is complete. Note that if $T_0 = \bar{L}'$, where L is a partial differential operator on G, then the eigenfunctions of the dual of T_0 are precisely the eigenfunctions of L, locally integrable solutions of $(L-\lambda I)u = 0$ with differentiation taken in the distribution sense.

Proof of Theorem 4. If every locally integrable solution of $(L-\lambda I)u = 0$, for a fixed λ, lies in $C^\infty(G)$, then the identity mapping J of the space of such solutions, with the inductive limit topology induced by the $L^1(C)$-norm for compact subsets C of G, into $C^\infty(G)$, is a closed linear mapping from an LF-space into an F-space. Applying the closed graph theorem, which is an easy consequence of the closed graph theorem for mappings of F-spaces, J is continuous. Thus, given a compact subset C of G and an integer $j\geqq 0$, there is a compact subset C_1 of G and a constant $k_{C,j}$ such that

$$|D^\alpha u(x)| \leqq \int_{C^1} |u(x)|\,dx, \qquad x\in C, \tag{24}$$

for every locally integrable weak solution u of $(L-\lambda I)u = 0$.

Consider the N-th partial sum $\psi_N(x, y, \lambda)$ of (11). Since by the hypothesis of the present theorem, $\phi_j(\cdot, \lambda)$ for each fixed λ is essentially in $C^\infty(G)$, we may assume that each $\phi_j(\cdot, \lambda)$ is actually in $C^\infty(G)$ by replacing its value at each point x of G by the limit of its average on a family of spheres shrinking down to the point. The resulting function differs from the original ϕ_j only on a null-set in $G \times R^1$. In particular, each ψ_N is infinitely differentiable on $G \times G$ for each fixed λ in R^1. By (24) and the finiteness a.e. of $q(x, \lambda)$, using the argument of the proof of Theorem 3, it follows that except on a m-null set π in R^1,

$$\sum_{j \leqq N} | D^\alpha \phi_j(x, \lambda) |^2 < \infty \tag{25}$$

uniformly, for each fixed λ in $R^1 - \pi$, for x in any compact subset C of G.

We may replace $\phi_j(x, \lambda)$ by zero on $G \times \pi$ without affecting any of its essential properties and assume that (25) holds for all λ. It follows by an elementary estimation that $D_x^\alpha D_x^\beta \psi_N(x, y, \lambda)$ converges for each fixed λ in R^1 and every α and β, uniformly on compact subsets of $G \times G$. Consequently, $\psi(x, y, \lambda)$, for fixed λ in R^1, lies in $C^\infty(G \times G)$, and the other conclusions of Theorem 4 follow.

The proof of Theorem 5 is essentially identical with that of Theorem 3, except that at the very beginning, the extension T_1 of T to the larger Hilbert space H_1 is normal, rather than self-adjoint. The only change that this necessitates is that all integrations in the spectral parameter must be taken over the complex plane instead of the real line R^1.

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NUMERICAL EQUIVALENCE AND THE ZETA-FUNCTION OF A VARIETY.*

By J. H. Sampson and G. Washnitzer.†

To Professor Zariski on his sixtieth birthday.

Introduction. One of the important problems of abstract algebraic geometry is to prove that there is a finite base for the group $\mathfrak{N}(V)$ of numerical equivalence classes of cycles on a variety V. It is known that the subgroup of $\mathfrak{N}(V)$ consisting of classes of divisors on V is finitely generated. That follows from the theorem of Néron-Severi [5], which shows that the group of algebraic equivalence classes of divisors on V is finitely generated. In characteristic zero it is a consequence of topological considerations that $\mathfrak{N}(V)$ is finitely generated; but in characteristic $p \neq 0$ very little is known in general about that group.

Given a non-singular projective variety V defined over a finite field with q elements, we establish here a connection between the theory of the base of $\mathfrak{N}(V \times V)$ and the Weil zeta-function $Z(u)$ of V: The assumption that $\mathfrak{N}(V \times V)$ (or a suitable part of it) is finitely generated permits us to deduce that $d \log Z(u)/du$ is a rational function of u. The proof is contained in §§ 1-5. It is closely bound up with the graphs in $V \times V$ of the q^r-th power mappings of V onto itself.

Assuming that $d \log Z(u)/du$ is rational, but without any reference to the theory of the base, we show in § 6 that this function has only simple poles. (To establish the rationality of $Z(u)$ one would have to show that the residues at those poles are rational integers.) The argument of § 6 is based on properties of power series with integral coefficients. It is possible to reach the same conclusion (in a more conceptual way) by an analysis of the algebraic structure of the ring of correspondences of V, again invoking the theory of the base. That argument will be presented at a later time.

In § 7 we verify Weil's conjectured functional equation for $Z(u)$. It is a straightforward consequence of the canonical involution of $V \times V$. Finally,

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in § 8 we sketch a sheaf-theoretic formulation of the argument of §§ 2-5. That formulation immediately sugegsts a number of generalizations, of which only bare mention is made here.

1. The zeta-function of a variety. Let V^r be a non-singular projective variety defined over a finite field K_1 containing q elements. For $\nu = 1, 2, 3$, etc. let K_ν denote the unique extension of K_1 in the universal domain containing q^ν elements. Let N_ν denote the number of points on V which are rational over K_ν, and let D_ν denote the number of positive zero-cycles of degree ν on V which are rational over K_1. We shall be concerned with the functions [1]

$$\Phi(u) = \sum_{\nu=1}^{\infty} N_\nu u^{\nu-1}, \qquad Z(u) = \sum_{\nu=0}^{\infty} D_\nu u^\nu \qquad (D_0 = 1). \tag{1}$$

They are related by the equation [2]

$$d \log Z(u)/du = \Phi(u). \tag{2}$$

2. Some preliminaries.

PROPOSITION 1. *Let U, W be non-singular projective varieties (complete), and let f be a regular rational transformation of U into W, with graph F in $U \times W$. Let X be a U-cycle and Y a W-cycle such that $X \times W, F, U \times Y$ intersect properly and such that $(X \times W) \cdot F$ and $F \cdot (U \times Y)$ are defined. Then* [3]

$$f(X \cdot f^{-1}(Y)) = f(X) \cdot Y. \tag{3}$$

Proof. Suppose first $\dim X \leqq \dim W$. The cycle $Q = (X \times W) \cdot F \cdot (U \times Y)$ is defined, and therefore $Z = X \cdot \mathrm{pr}_U[F \cdot (U \times Y)]$ is defined and equal to $\mathrm{pr}_U(Q)$ [F-VII, Th. 10, 16]. By definition, $Z = X \cdot f^{-1}(Y)$. Further, $(Z \times W) \cdot F$ is defined and equal to Q [F-VII, Th. 17, Cor. 3]. Then $f(X \cdot f^{-1}(Y)) = f(Z) = \mathrm{pr}_W(Q) = \mathrm{pr}_W\{[F \cdot (X \times W)] \cdot (U \times Y)\}$, which in turn is $\mathrm{pr}_W[F \cdot (X \times W)] \cdot Y = f(X) \cdot Y$ [F-VII, Th. 16]. If $\dim X > \dim W$, then both sides of (3) must be zero, from dimensional considerations.

Remark. In calculations with rational, algebraic or numerical equivalence

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[1] $\Phi(u)$ and $Z(u)$ were introduced by Weil in C. A., II• partie, §§ IV-V, for curves, and in [11] for arbitrary dimension r. See also V. A., § IX, No. 69 and [12].

[2] Proved for curves in C. A., II• partie, Nos. 18, 19, 20. That demonstration holds *mutatis mutandis* for any dimension r.

[3] Notation essentially as in V. A., § 1, No. 3: $f(X) = \mathrm{pr}_W[F \cdot (X \times W)]$ and $f^{-1}(Y) = \mathrm{pr}_U[F \cdot (U \times Y)]$, etc.

classes of cycles it is always possible to find representative cycles for which the conditions of Proposition 1 are fulfilled, in virtue of results of W.-L. Chow [2, Lemma 2, Corollary 4, § 4 and Lemma 4, § 6]. Hence formula (3) is of unrestricted validity when applied to such equivalence classes.

PROPOSITION 2. *Let U, W, f be as in Proposition* 1, *with f now assumed purely inseparable and one-to-one. Let Y be a subvariety of W. Then the cycle $f^{-1}(Y)$ is always defined. K being a field of definition for f and Y, let y be a generic point of Y over K, and let y' be the unique point of U such that $f(y') = y$. If K is perfect, then y' has a locus Y' over K, and $f^{-1}(Y) = nY'$, where $n = (\deg f)/[K(y') : K(y)]$.*

Proof. F being the graph of f, one easily sees that every point of $F \cap (U \times Y)$ must be a specialization of (y', y) over K. Hence the first point. Since $f(y') = y$, we have $K(y', y) = K(y')$, which, by assumption, is a purely inseparable extension of $K(y)$. Consequently the ideal determined by y' over the algebraic closure of K has a basis with coefficients in a purely inseparable extension of K. If K is perfect, then $K(y')$ must be a regular extension. Thus (y', y) has a locus over K, which must be precisely $F \cap (U \times Y)$; and y' has a locus over K which is simply the projection of $F \cap (U \times Y)$ on U. Therefore $f^{-1}(Y) = \mathrm{pr}_U[F \cdot (U \times Y)] = nY'$, where n is the intersection multiplicity of F, $U \times Y$.

To calculate n, let Z be any subvariety of U. Then the cycle $f(Z) = \mathrm{pr}_W[F \cdot (Z \times W)]$ is always defined, clearly. The cycle $Z' = F \cdot (Z \times W)$ is a variety [F-VII, Th. 17]. Denote its projection on W by Z''. Then by definition $f(Z) = [Z' : Z'']Z''$. Taking $Z = Y'$ we see that Z' is the locus of (y', y) over K. Consequently $f(Y') = [K(y') : K(y)] \cdot Y$. But from formula (3), with $X = U$, we have $f(f^{-1}(Y)) = f(U) \cdot Y = (\deg f) \cdot Y$, whence $n \cdot f(Y') = (\deg f) \cdot Y$, or finally $n \cdot [K(y') : K(y)] = \deg f$.

3. The transformation φ. Returning now to the variety of § 1, we introduce the mapping ϕ of $V \times V$ onto itself defined for any point (u, v) of $V \times V$ by $\phi(u, v) = (u^q, v)$, where $u \to u^q$ is the q-th power mapping of V onto itself.[4] I.e., ϕ is the product of the q-th power mapping of V on the first factor of $V \times V$ and of the identity mapping on the second factor. ϕ is a regular rational one-to-one mapping of $V \times V$ onto itself, defined over K_1; it has degree q^r. Denoting the ν-th iterate of ϕ by ϕ^ν, we have $\phi^\nu(u, v) = (u^{q^\nu}, v)$.

[4] If V is a subvariety of a projective space P^m, with a specified system of homogeneous coordinates in P^m, and if the point u has coordinates $u_0, \cdots, u_m$, then u^q denotes the point with coordinates $u_0^q, \cdots, u_m^q$. Similarly, u^{q^ν} has coordinates $u_0^{q^\nu}, \cdots, u_m^{q^\nu}$ ($\nu = 1, 2, 3$, etc.). Cf. F-VII, § 2.

In addition to ϕ we shall require the mapping σ of $V \times V$ defined by $\sigma(u, v) = (v, u)$ and also the q-th power mapping of $V \times V$ onto itself, denoted by τ: $\tau(u, v) = (u^q, v^q)$. Both σ and τ are purely inseparable regular rational transformations defined over K_1, and of course σ is biregular. ϕ, σ and τ are related by the identities

$$\sigma\sigma = \mathbf{1}, \qquad \sigma\phi\sigma\phi = \tau, \tag{4}$$

and more generally

$$\sigma\phi^\nu\sigma\phi^\nu = \tau^\nu, \tag{4'}$$

where $\mathbf{1}$ is the identity mapping of $V \times V$ and where ϕ^ν, τ^ν are the ν-fold iterates of ϕ, τ.

4. The diagonal cycle. Now let Δ_0 denote the diagonal in $V \times V$. From Proposition 2 it follows at once that the cycle[5] $\Delta_\nu = \phi^{-\nu}(\Delta_0)$ is defined and is a subvariety of $V \times V$ having K_1 as a field of definition ($\nu = 1, 2, 3,$ etc.). It is moreover a non-singular subvariety of $V \times V$, as follows from the Jacobian criterion. The intersection set $\Delta_\nu \cap \Delta_0$ ($\nu > 0$) consists precisely of the points of Δ_0 which are rational over K_ν. The number of those points is therefore N_ν (cf. § 1 above). It is quickly seen that Δ_ν and Δ_0 are transversal at each point of intersection ($\nu > 0$). Therefore

$$N_\nu = \deg(\Delta_\nu \cdot \Delta_0) \qquad (\nu = 1, 2, 3, \cdots). \tag{5}$$

Now let $\mathfrak{N}(V \times V)$ denote the group of numerical equivalence classes of cycles on $V \times V$ (cf. Weil, F-IX, § 7); $\mathfrak{N}^r(V \times V)$ will stand for the subgroup consisting of classes of dimension r. Let $\mathfrak{d}_\nu$ denote the numerical equivalence class of the cycle Δ_ν ($\nu = 0, 1, 2, \cdots$). The $\mathfrak{d}_\nu$ are of course elements of $\mathfrak{N}^r(V \times V)$. Indicating the canonical scalar product in $\mathfrak{N}^r(V \times V)$ by the symbol $\langle \mathfrak{x}, \mathfrak{y} \rangle$, we have from (5)

$$N_\nu = \langle \mathfrak{d}_\nu, \mathfrak{d}_0 \rangle \qquad (\nu = 1, 2, 3, \cdots). \tag{6}$$

Each of the mappings ϕ^ν, σ, τ^ν induces an endomorphism of $\mathfrak{N}(V \times V)$ which preserves codimension. Those endomorphisms will be called $\phi^{\nu *}$, σ^*, $\tau^{\nu *}$ respectively. To describe them unambiguously, consider the mapping ϕ: Let $\mathfrak{x}$ be any numerical equivalence class, X a representative cycle. Then $\phi^{-1}(X)$ is defined, by Proposition 2, and by definition $\phi^*(\mathfrak{x})$ denotes its numerical equivalence class. Analogous definitions for $\phi^{\nu *}$, σ^*, $\tau^{\nu *}$. (In this

[5] $\phi^{-\nu}(\Delta_0)$ naturally stands for $(\phi^\nu)^{-1}(\Delta_0)$. We mention that, for the immediate purpose, we could equally well use the direct image cycle $\phi^\nu(\Delta_0)$. From § 2 above one easily verifies that $\phi^{-\nu}(\Delta_0) = \sigma(\phi^\nu(\Delta_0))$.

connection see Chow [2], § 6, especially Theorem 4.) It is clear from the definitions that

$$\mathfrak{d}_\nu = \phi^{\nu *}(\mathfrak{d}_0) \tag{7}$$

and

$$\phi^{\nu *} = (\phi^*)^\nu \qquad (\nu\text{-fold iterate of } \phi^*). \tag{8}$$

Hence

$$N_\nu = \langle \phi^{\nu *}(\mathfrak{d}_0), \mathfrak{d}_0 \rangle = \langle (\phi^*)^\nu (\mathfrak{d}_0), \mathfrak{d}_0 \rangle. \tag{9}$$

Remark. In the expressions above it will be convenient to allow ν to take on the value zero. By ϕ^{0*} or $(\phi^*)^0$ we understand the identity endomorphism of $\mathfrak{N}(V \times V)$. The number N_0 defined by (6) or (9) for $\nu = 0$ is the self-intersection number of the diagonal Δ_0, hence the Euler-Poincaré characteristic of V. We shall also denote it by $\chi(V)$.

5. The rationality of $\mathfrak{P}(u)$. As explained in the Introduction, our results concerning the zeta-function of V are based on a certain assumption concerning the group $\mathfrak{N}(V \times V)$. We now introduce that assumption explicitly. Write $\mathfrak{d}'_\nu = \sigma^*(\mathfrak{d}_\nu)$ $(\nu = 1, 2, \cdots)$, where, as above, σ^* is the involution of $\mathfrak{N}(V \times V)$ induced by the mapping σ. Of course $\mathfrak{d}'_0 = \mathfrak{d}_0$. Let G denote the subgroup of $\mathfrak{N}^r(V \times V)$ generated by the classes[6] $\mathfrak{d}_\nu$ and $\mathfrak{d}'_\nu$ $(\nu = 0, 1, 2, \cdots)$.

HYPOTHESIS. *It will henceforth be supposed that the group G is finitely generated.*[7]

Consider now the mapping τ^μ, which has degree $q^{2r\mu}$. Applying Proposition 2 we find at once that $\tau^{-\mu}(\Delta_\nu) = q^{r\mu}\Delta_\nu$. Similarly, writing $\Delta'_\nu = \sigma^{-1}(\Delta_\nu)$ (so that Δ'_ν is a representative cycle of $\mathfrak{d}'_\nu$), we have $\tau^{-\mu}(\Delta'_\nu) = q^{r\mu}\Delta'_\nu$. Therefore

$$\tau^{\mu *} = (\tau^*)^\mu = q^{r\mu} \times \textit{identity in } G. \tag{10}$$

Again from Proposition 2 we obtain $\phi^{-1}(\Delta'_\nu) = q^r \Delta'_{\nu-1}$ $(\nu > 0)$; and of course $\phi^{-1}(\Delta_\nu) = \Delta_{\nu+1}$. Therefore ϕ^* maps G into itself. Since $\sigma^*\sigma^* = \text{identity}$, σ^* also maps G into itself. In what follows we shall be concerned solely with the action of ϕ^* and σ^* in G.

The group $\mathfrak{N}(V \times V)$ is free from torsion. Therefore, because of our hypothesis, G must be a free group, say of rank ρ. The endomorphism ϕ^*

[6] The classes $\mathfrak{d}'_\nu$ are included in G to ensure that σ^* maps G into itself. They will not play an essential rôle until No. 7 below.

[7] For $r = 1$ the finiteness is an immediate consequence of the Severi-Néron theory of the base (Néron [5], § 11, Theorem 2).

must consequently satisfy its characteristic equation. I. e., there exist rational integers $e_1, \cdots, e_\rho$ such that

$$(\phi^*)^\nu + e_1(\phi^*)^{\nu-1} + \cdots + e_\rho(\phi^*)^{\nu-\rho} = 0 \text{ in } G \tag{11}$$

for every $\nu \geqq \rho$.[8] From (9) we obtain

$$N_\nu + e_1 N_{\nu-1} + \cdots + e_\rho N_{\nu-\rho} = 0 \qquad (\nu \geqq \rho), \tag{12}$$

which is a linear difference equation for the numbers N_ν $(\nu \geqq 0)$. The rationality of the function $\Phi(u)$ defined by (1) is an immediate consequence of (12).

6. The functional form of $\Phi(u)$. We now show that the rationality of $\Phi(u)$ implies that it has simple poles. The argument of this section is based on certain properties of power series with integral coefficients; it is independent of the finiteness assumption made in § 5 above.

Let us temporarily write $\Phi_1(u)$ for $\Phi(u)$. If we now replace the ground field K_1 by one of the extensions K_h, then $\Phi_1(u)$ is replaced by the corresponding function (cf. § 1)

$$\Phi_h(u) = \sum_{\nu=1}^{\infty} N_{\nu h} u^{\nu-1} \qquad (h = 1, 2, 3, \cdots).$$

From (2) it follows that $\exp\{\int_0^u \Phi_h(u)\,du\}$ is a power series with *integral* coefficients, being the zeta-function of the variety V with reference to K_h. The hypotheses of the following theorem are therefore satisfied by $\Phi(u)$:

THEOREM 1. *Let $R(x) = \sum_{\nu=1}^{\infty} a_\nu x^{\nu-1}$ be a power series with integral coefficients which represents a rational function of x. For $h = 1, 2, 3$, etc. set $R_h(x) = \sum_{\nu=1}^{\infty} a_{\nu h} x^{\nu-1}$, and suppose that for each h the function $\exp\{\int_0^x R_h(x)dx\}$ has a representation as a power series in x with integral coefficients. Then $R(x)$ has a partial fraction decomposition of the form*

$$R(x) = \gamma_1/(1-\alpha_1 x) + \cdots + \gamma_s/(1-\alpha_s x),$$

where $\alpha_1, \cdots, \alpha_s$ and $\gamma_1, \cdots, \gamma_s$ are algebraic numbers. The α_i must in fact be algebraic integers.

Proof. First we recall that the rationality of $R(x) = R_1(x)$ implies that the coefficients a_ν satisfy a linear difference equation *for all large* ν.

[8] One could equally well use the minimal equation of ϕ^* in G.

I.e., there exists a relation of the form $c_0 a_\nu + \cdots + c_\rho a_{\nu-\rho} = 0$ for all large ν, where $c_0, c_1, \cdots, c_\rho$ are fixed rational integers. (We assume that ρ has the minimum possible value.) If the c_j are relatively prime and $c_0 > 0$, then it is rather easily seen that $c_0 = 1$, since the a_ν are all integers. Hence, our relation has the form $a_\nu + c_1 a_{\nu-1} + \cdots + c_\rho a_{\nu-\rho} = 0$. Let $\alpha_1, \cdots, \alpha_s$ be the distinct roots of the equation $\xi^\rho + c_1 \xi^{\rho-1} + \cdots + c_\rho = 0$. The α_i are thus algebraic integers. Now by solving the difference equation above (for large ν), we can reconstruct the function $R(x)$, apart from a polynomial. It is readily seen that $R(x)$ must have a partial fraction decomposition of the form

$$R(x) = \sum \gamma_j/(1-\alpha_j x)^{m_j} + P(x), \tag{13}$$

where the α_j are as above (but possibly repeated for several exponents m_j), where the γ_j are algebraic numbers, and where $P(x)$ is a polynomial with integral coefficients.

Consider now the integral power series

$$\exp\{\int_0^x R(x)\,dx\} = \exp\{\sum_{\nu=1}^{\infty} (a_\nu/\nu)x^\nu\} = \prod_{\nu=1}^{\infty} \exp\{(a_\nu/\nu)x^\nu\}.$$

Let p be any prime. From the infinite product expansion it is easily seen that the coefficient of x^p in the power series expansion is $(a_1{}^p/p!) + (a_p/p) + Q$, where Q is a rational number whose denominator does not contain p. Then, since $p!Q$ is an integer, so is $(p-1)!Q$. Consequently, $a_1{}^p + (p-1)!\,a_p \equiv 0 \pmod p$, whence $a_1 \equiv a_p \pmod p$ (congruences of Wilson and Fermat). The same argument applied to $R_h(x)$ instead of $R_1(x)$ yields

$$a_h \equiv a_{hp} \pmod p \tag{14}$$

for all primes p and all $h = 1, 2, 3$, etc.

We now compute the coefficients of x^{h-1} and x^{hp-1} from (13). For x^{h-1} we obtain

$$\sum \gamma_j \alpha_j{}^{h-1} m_j(m_j+1)\cdots(m_j+h-2)/(h-1)! + b_{h-1}, \tag{15}$$

where b_{h-1} is the coefficient of x^{h-1} in the polynomial $P(x)$. For all large p the coefficient of x^{hp-1} is

$$\sum \gamma_j \alpha_j{}^{hp-1} m_j(m_j+1)\cdots(m_j+hp-2)/(hp-1)!. \tag{16}$$

Now, for $h < p$, $(hp-1)!$ contains the factor p^{h-1} but not p^h. If $1 < m < p$, then the product $m(m+1)\cdots(m+hp-2)$ contains the factor p^h. Thus

$$m_j(m_j+1)\cdots(m_j+hp-2)/(hp-1)! \equiv 0 \pmod p \text{ for } p \gg 0,\ m_j \neq 1. \tag{17}$$

Let E be the number field generated by the numbers α_i, γ_j. Let $\mathfrak{p}$ be a prime ideal in E of degree 1 with associated rational prime p, assumed large. By Fermat's theorem we have $\alpha_j^{hp} = (\alpha_j^h)^{N\mathfrak{p}} \equiv \alpha_j^h \pmod{\mathfrak{p}}$. Hence from (16), which is equal to a_{hp} for large p, we obtain

$$a_{hp} \equiv \sum_{(m_j=1)} \gamma_j \alpha_j^{h-1} \pmod{\mathfrak{p}}$$

for all primes $\mathfrak{p}$ of degree 1 with large $N\mathfrak{p} = p$ (the sum is to be extended over all terms for which $m_j = 1$). From this fact and from (14), (15) there follows

$$\sum_{(m_j>1)} \gamma_j \alpha_j^{h-1} m_j(m_j+1) \cdots (m_j+h-2)/(h-1)! + b_{h-1} \equiv 0 \pmod{\mathfrak{p}}, \tag{18}$$

the sum being over all terms for which $m_j > 1$. Since this congruence holds for infinitely many primes $\mathfrak{p}$,[9] we conclude that the left member of (18) is equal to zero for all h. But the left member of (18) is precisely the contribution to the coefficient a_h of x^{h-1} in $R(x)$ from the terms

$$\sum_{(m_j>1)} \gamma_j/(1-\alpha_j x)^{m_j} + P(x) \tag{19}$$

in the partial fraction decomposition (13). The contribution of (19) to $R(x)$ is therefore zero, which proves Theorem 1.

Solving (12) for the N_ν and applying Theorem 1, we find that $\Phi(u)$ must have the form

$$\Phi(u) = \beta_1\alpha_1/(1-\alpha_1 u) + \cdots + \beta_t\alpha_t/(1-\alpha_t u), \tag{20}$$

where now $\alpha_1, \cdots, \alpha_t$ are the *distinct* roots of the equation

$$x^\rho + e_1 x^{\rho-1} + \cdots + e_\rho = 0, \tag{21}$$

the e_j being as in (11) and (12). The β_j are algebraic numbers which must satisfy the relation

$$N_\nu = \sum_{j=1}^{t} \beta_j \alpha_j^\nu \tag{22}$$

for all $\nu = 0, 1, 2$, etc. We note in particular that

$$\chi(V) = N_0 = \beta_1 + \cdots + \beta_t. \tag{23}$$

The zeta-function itself must have the form

$$Z(u) = \prod_{j=1}^{t} (1-\alpha_j u)^{-\beta_j}, \tag{24}$$

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[9] See e. g., Hecke [3], Satz. 126, p. 165.

suitable branches being chosen if the β_j are not rational integers. They presumably are. We note that

$$N_\nu \equiv N_{\nu p} \pmod{p} \tag{25}$$

for $\nu = 1, 2, 3$, etc. and all rational primes p, as follows from (14).

7. The functional equation of $Z(u)$. We return now to the group G defined in § 5 and to the endomorphisms ϕ^* and σ^*. From (4) and (10) there follow the relations

$$\begin{cases} \sigma^*\sigma^* = \textit{identity} \\ \phi^*\sigma^*\phi^*\sigma^* = q^r \times \textit{identity} \end{cases} \quad \textit{in } G. \tag{26}$$

The calculations we are about to make are somewhat more perspicuous in terms of matrices. Let us therefore choose a base $\mathfrak{g}_1, \cdots, \mathfrak{g}_\rho$ for G and put

$$\begin{cases} \phi^*(\mathfrak{g}_i) = \sum_{j=1}^{\rho} a_{ij}\mathfrak{g}_j \\ \sigma^*(\mathfrak{g}_i) = \sum_{j=1}^{\rho} s_{ij}\mathfrak{g}_j \end{cases} \qquad (i = 1, \cdots, \rho), \tag{27}$$

the a_{ij} and s_{ij} being rational integers. Write $A = (a_{ij})$, $S = (s_{ij})$. The relations (26) then become

$$\begin{cases} S^2 = I \\ (SA)^2 = q^r I \end{cases} \qquad (I = \rho \times \rho \text{ unit matrix}). \tag{28}$$

From this one concludes easily

$$(SA^\nu)^2 = q^{r\nu} I \qquad (\nu = 0, 1, 2, \cdots), \tag{28'}$$

which is just (4') in G.

Remark. As mentioned earlier, for curves ($r = 1$) the group $\mathfrak{N}^1(V \times V)$ is finitely generated. If in this case we take for G the entire group $\mathfrak{N}^1(V \times V)$, then our preceding calculations remain valid, except possibly (10), since $\mathfrak{N}^1(V \times V)$ may not possess a base consisting of classes which are rational over K. If it does, however, then from (28) there results $(\det A)^2 = q^\rho$, where ρ is the Picard number of $V^1 \times V^1$. Since A is an integral matrix, ρ must be even if q is an odd power of a prime. *Then whether or not $V^1 \times V^1$ has a base for divisors rational over K_1, it follows that $\rho > 3$ whenever q is an odd power of a prime.*

Resuming the general argument, we wish to consider the characteristic polynomial $f(x)$ of the matrix A. Of course $f(x) = x^\rho + e_1 x^{\rho-1} + \cdots + e_\rho$,

the coefficients e_j being as in (11), (12), (21). We have $f(x) = \det(xI - A)$ $= \det[S(xI - A)S]$, since $(\det S)^2 = 1$. Hence

$$f(x) = \det(xI - SAS) = \det(xI - q^r A^{-1})$$
$$= (-1)(\det A)^{-1} x^\rho \det((q^r/x)I - A),$$

by (28). Thus

$$f(x) = e_\rho^{-1} x^\rho f(q^r/x). \tag{29}$$

Again let $\alpha_1, \cdots, \alpha_t$ be the distinct roots of (21)—i.e., the distinct roots of $f(x)$. For each α_i the quantity q^r/α_i must also be a root of $f(x)$, by (29).[10] Call it α_{si}, so that $i \to si$ designates a permutation of $1, \cdots, t$ of period 2.

Referring now to the numbers $\beta_1, \cdots, \beta_t$ introduced in (20), (22), it is important to show that $\beta_{si} = \beta_i$ for each i. To prove this, write the diagonal class $\mathfrak{d}_0$ in terms of the base $\mathfrak{g}_1, \cdots, \mathfrak{g}_\rho$. Say $\mathfrak{d}_0 = c_1\mathfrak{g}_1 + \cdots + c_\rho\mathfrak{g}_\rho$. Since $\sigma^*(\mathfrak{d}_0) = \mathfrak{d}_0$, we have

$$\sum_{i=1}^{\rho} c_i s_{ij} = c_j \qquad (j = 1, \cdots, \rho). \tag{30}$$

Let the coefficients of A^ν be denoted by $a_{ij}^{(\nu)}$. Then from (9) we obtain

$$N_\nu = \langle \mathfrak{d}_\nu, \mathfrak{d}_0 \rangle = \sum_{i,j,k} c_i a_{ij}^{(\nu)} c_k \langle \mathfrak{g}_j, \mathfrak{g}_k \rangle. \tag{31}$$

Since σ is a biregular birational transformation it is clear that

$$N_\nu = \langle \mathfrak{d}_\nu, \mathfrak{d}_0 \rangle = \langle \sigma^*(\mathfrak{d}_\nu), \sigma^*(\mathfrak{d}_0) \rangle = \langle \mathfrak{d}'_\nu, \mathfrak{d}_0 \rangle,$$

and so

$$N_\nu = \sum c_i a_{ij}^{(\nu)} s_{jk} c_k \langle \mathfrak{g}_h, \mathfrak{g}_k \rangle.$$

From (28′), $A^\nu S = q^{r\nu} S A^{-\nu}$. Therefore the last equation can be written

$$N_\nu = q^{r\nu} \sum c_i s_{ij} a_{jk}^{(-\nu)} c_h \langle \mathfrak{g}_h, \mathfrak{g}_k \rangle.$$

Using (30) we obtain

$$N_\nu = q^{r\nu} \sum c_i a_{ij}^{(-\nu)} c_k \langle \mathfrak{g}_j, \mathfrak{g}_k \rangle. \tag{32}$$

Let us now define N_ν for $\nu < 0$ by means of the difference equation (12). It is clear that the values so obtained for N_{-1}, N_{-2}, etc. must be the same as the values calculated from (31) by putting $\nu = -1, -2$, etc. Referring to (32) we conclude that

$$N_\nu = q^{r\nu} N_{-\nu} \qquad (\nu = 0, 1, 2, \text{etc.}) \tag{33}$$

Now from (22) we have (for any value of ν)

[10] Each α_i is thus an algebraic integer that divides some power of the field characteristic.

$$N_\nu = \sum_{i=1}^{t} \beta_i \alpha_i^\nu = \sum \beta_i q^{r\nu}/\alpha_{si}^\nu = q^{r\nu} \sum \beta_i \alpha_{si}^{-\nu}.$$

From (33),

$$N_\nu = q^{r\nu} \sum \beta_i \alpha_i^{-\nu}.$$

Therefore, $\sum(\beta_i - \beta_{si})\alpha_i^{-\nu} = 0$ for all ν. Since the α_i are distinct, we conclude that $\beta_i = \beta_{si}$. Now from (20),

$$\Phi(u) = \sum_{i=1}^{t} \beta_i \alpha_i/(1 - \alpha_i u),$$

Hence

$$\Phi(1/q^r u) = \sum \beta_i \alpha_i/(1 - \alpha_i/q^r u)$$
$$= \sum \beta_i \alpha_i/(1 - 1/\alpha_{si} u) = -\sum \beta_i \alpha_i \alpha_{si} u/(1 - \alpha_{si} u).$$

Since $\alpha_i \alpha_{si} = q^r$, $\beta_i = \beta_{si}$, this becomes

$$\Phi(1/q^r u) = -q^r u \sum \beta_i/(1 - \alpha_i u)$$
$$= -q^r u^2 \sum \beta_i \alpha_i/(1 - \alpha_i u) - q^r u \sum \beta_i.$$

From (23) we have $\sum \beta_i = \chi(V)$. Hence

$$\Phi(1/q^r u) = -q^r u^2 \cdot \Phi(u) - q^r u \cdot \chi(V). \tag{34}$$

Let us now assume that the β_i are rational integers, which of course must be the case if $Z(u)$ is a rational function. From (24) and the preceding calculations we derive the functional equation

$$Z(1/q^r u) = (-1)^{\chi(V)} (\prod \alpha_i^{\beta_i}) \cdot u^{\chi(V)} Z(u). \tag{35}$$

If no α_i is equal to $q^{r/2}$, then by putting $u = q^{-r/2}$ in (35) we find $\prod \alpha_i^{\beta_i} = (-1)^\chi q^{r\chi/2}$ $(\chi = \chi(V))$. Moreover in this case $\chi(V)$ must be even, because the β_i occur in equal pairs. It is easy to see that even if some α_i is equal to $q^{r/2}$, say $\alpha_1 = q^{r/2}$, we must nonetheless have $\prod \alpha_i^{\beta_i} = q^{r\chi/2}$, if we continue with our assumption that the β_i are rational integers. For $\alpha_2, \cdots, \alpha_t$ must occur in pairs α_i, α_{si}, and we have $\alpha_i^{\beta_i} \alpha_{si}^{\beta_{si}} = (\alpha_i \alpha_{si})^{\beta_i} = q^{r\beta_i}$. The assertion follows easily. Hence, if the β_i are integers we obtain the following equation for $Z(u)$:

$$Z(1/q^r u) = (-1)^\chi q^{r\chi/2} u^\chi \cdot Z(u) \qquad (\chi = \chi(V)). \tag{36}$$

This agrees precisely with the form predicted by Weil [11], save that Weil's factor ± 1 is here found to be $(-1)^\chi$.

8. **A sheaf-theoretic formulation.** In the framework of sheaf theory

the foregoing discussion assumes a rather different aspect, of which we shall now give a brief account. Needless to say, the finiteness assumption made in § 5 cannot be circumvented. It takes on here a slight different form, relating to the Chern classes of the sheaves involved. In order to avail ourselves of Serre's sheaf theory we shall now take as ground field the algebraic closure K of the field K_1. All functions are supposed to be rational over K.

Let $\mathcal{O}$ denote the sheaf of local rings on $V \times V$, and let $\mathcal{O}_\nu$ denote the sheaf of local rings of the subvariety Δ_ν, extended by zero to all of $V \times V$. From results of Serre [7, No. 55, Lemme 2, Corollaire] and from standard theorems of homological algebra (Cartan-Eilenberg [1], Proposition 4.3, p. 151; Theorem 6.1′, p. 157) it follows that there exists a *resolution* of $\mathcal{O}_0$ $(= \mathcal{O}_{\Delta_0})$ consisting of locally free sheaves $\mathcal{E}_j$ on $V \times V$ $(j = 1, \cdots, r)$ and homomorphisms such that the sequence

$$0 \to \mathcal{E}_r \to \mathcal{E}_{r-1} \to \cdots \to \mathcal{E}_1 \to \mathcal{O} \to \mathcal{O}_0 \to 0 \tag{37}$$

is exact. In fact, for $j < r$, $\mathcal{E}_j$ can be taken to be a direct sum of locally free sheaves of dimension 1.

Now for any coherent sheaf $\mathfrak{F}$ on $V \times V$ we can form the *reciprocal image* sheaf $\mathfrak{F}^{(\nu)} = \phi^{-\nu}(\mathfrak{F})$ with respect to the regular mapping ϕ^ν. Viz., for any point (u, v) write $\phi^\nu(u, v) = (u', v)$, $u' = u^{q^\nu}$. Then the stalk of $\mathfrak{F}^{(\nu)}$ at (u, v) is defined to be $\mathcal{O}_{(u,v)} \otimes_{\phi^\nu} \mathfrak{F}_{(u',v)}$, meaning the tensor product over $\mathcal{O}_{(u',v)}$, where $\mathcal{O}_{(u,v)}$ is regarded as $\mathcal{O}_{(u',v)}$-module *via* ϕ^ν. This operation is fully described in [6]. We have $\phi^{-\nu}(\mathcal{O}) = \mathcal{O}$, and it is readily seen that $\mathcal{E}_1 \to \mathcal{O}$ induces a homomorphism $\phi^{-\nu}(\mathcal{E}_1) \to \phi^{-\nu}(\mathcal{O})$, i.e., $\mathcal{E}_1{}^{(\nu)} \to \mathcal{O}$, with the property that the image of $\mathcal{E}_1{}^{(\nu)}$ in $\mathcal{O}$ is precisely the sheaf of ideals of the subvariety Δ_ν. Thus by taking reciprocal images of (37) we obtain for each ν a sequence

$$0 \to \mathcal{E}_r{}^{(\nu)} \to \mathcal{E}_{r-1}{}^{(\nu)} \to \cdots \to \mathcal{E}_1{}^{(\nu)} \to \mathcal{O} \to \mathcal{O}_\nu \to 0. \tag{38}$$

This sequence is exact. The proof can be outlined as follows (we take $\nu = 1$ for simplicity of notation): At any point (u, u) of Δ_0 we can find generators $\xi_1, \cdots, \xi_r$ for the ideal p of Δ_0 in $\mathcal{O}_{(u,u)}$ which form a subset of a regular system of parameters at (u, u) and which are such that their transforms $\xi_1' = \phi^*(\xi_1), \cdots, \xi_r' = \phi^*(\xi_r)$ by ϕ form a subset of a regular system of parameters at $(u^{1/q}, u)$ and generate the ideal p' of Δ_1 in $\mathcal{O}_{(u^{1/q},u)}$. The corresponding Koszul resolutions of p and p' are exact (Cartan-Eilenberg [1], Proposition 4.3, p. 151), and it is readily seen that the reciprocal image operation ϕ^{-1} simply transforms the Koszul resolution of p into the Koszul resolution of p'. Thus we conclude that $\mathrm{Tor}_h(\mathcal{O}, \mathcal{O}_0) = 0$ for $h \geqq 1$, where

again we mean that $\mathfrak{O}$ at $(u^{1/q}, u)$ and $\mathfrak{O}_0$ at (u, u) are both regarded as $\mathfrak{O}_{(u,u)}$-modules, the former by means of ϕ. The exactness of (38) then follows at once from the definition of the torsion functor.

Now from $\mathfrak{O}_\nu$ and $\mathfrak{O}_0$ form the ordinary tensor product over $\mathfrak{O}$ $(\nu > 0)$. At any of the N_ν points on $\Delta_\nu \cap \Delta_0$ the stalk of this sheaf is isomorphic to K, because Δ_ν and Δ_0 are transversal at each point of intersection. Therefore we have

$$N_\nu = \chi(\mathfrak{O}_\nu \otimes \mathfrak{O}_0) \qquad (\nu = 1, 2, 3, \cdots), \tag{39}$$

where for any coherent sheaf $\mathfrak{F}$ on $V \times V$ we understand

$$\chi(\mathfrak{F}) = \sum_{m=0}^{\infty} (-1)^m \dim_K H^m(V \times V, \mathfrak{F}).$$

We require now the fact that $\operatorname{Tor}_h^{\mathfrak{O}}(\mathfrak{O}_\nu, \mathfrak{O}_0) = 0$ for $h > 0$. This is easily established from the transversality of Δ_ν, Δ_0 by means of a Koszul resolution at any point of intersection. Consequently the tensor product, over $\mathfrak{O}$, of the two complexes (37), (38) is an acyclic resolution of $\mathfrak{O}_\nu \otimes \mathfrak{O}_0$. From the familiar properties of the Euler-Poincaré characteristic we arrive at the formula

$$N_\nu = \sum_{i,j=0}^{r} (-1)^{i+j} \chi(\mathcal{E}_i^{(\nu)} \otimes_{\mathfrak{O}} \mathcal{E}_j) \qquad (\nu > 0), \tag{40}$$

where we understand $\mathcal{E}_0^{(\nu)} = \mathcal{E}_0 = \mathfrak{O}$. We can then write

$$\Phi(u) = \sum_{\nu=1}^{\infty} N_\nu u^{\nu-1} = \sum_{i,j=1}^{r} \Phi_{ij}(u),$$

where

$$\Phi_{ij}(u) = (-1)^{i+j} \sum_{\nu=1}^{\infty} \chi(V \times V, \mathcal{E}_i^{(\nu)} \otimes_{\mathfrak{O}} \mathcal{E}_j) \cdot u^{\nu-1}. \tag{41}$$

Under the assumption of a finite base for numerical equivalence it is possible to prove a general result which yields the rationality of the above functions $\Phi_{ij}(u)$ as a special case: Let X be a non-singular projective variety (over a field of arbitrary characteristic); let $\psi: X \to X$ be a regular mapping of X into itself; and let $\mathfrak{F}$, $\mathfrak{G}$ be locally free sheaves defined on X. Consider the function

$$\Lambda(u) = \sum_{\nu=1}^{\infty} \chi(X, \mathfrak{F}^{(\nu)} \otimes \mathfrak{G}) \cdot u^{\nu-1},$$

where $\mathfrak{F}^{(\nu)}$ is the reciprocal image of $\mathfrak{F}$ with respect to the ν-th iterate ψ^ν of ψ. Then, under the assumption of a finite base, it can be proved that $\Lambda(u)$ is a rational function of u. This follows quickly with the aid of Hirzebruch's

formula for the Euler-Poincaré characteristic of a locally free sheaf.[11] It is probable that appropriate generalizations of Weil's conjectures could be made for $\Lambda(u)$.

Finally, there are some special cases where the rationality of $\Lambda(u)$ depends merely upon the existence of a finite base for numerical equivalence of divisors on X—for example, when either $\mathfrak{F}$ or $\mathfrak{G}$ is the sheaf of cross sections of a vector bundle whose structure group can be reduced to the group of triangular matrices.

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ON RAMIFICATION THEORY IN NOETHERIAN RINGS.*[1]

By M. AUSLANDER and D. A. BUCHSBAUM.

Introduction. The purpose of this paper is to give a fairly general ramification theory for noetherian rings. To illustrate the types of results obtained let us assume that S is a noetherian ring and R is a subring of S such that the kernel $\mathfrak{J}$ of the mapping $\phi: S \otimes_R S \to S$ defined by $\phi(x \otimes y) = xy$ is a finitely generated ideal in $S \otimes_R S$. In § 2 we show that S is an unramified extension of R (see § 1 for definition) if and only if S is a projectice $S \otimes_R S$-module, or equivalently, if and only if $\mathfrak{J}$ is a direct summand of $S \otimes_R S$. From this it follows that if $\mathfrak{H}_{S/R} = \phi(\mathfrak{N})$, where $\mathfrak{N}$ is the annihilator of $\mathfrak{J}$ in $S \otimes S$, then a prime ideal $\mathfrak{P}$ in S is ramified if and only if it contains $\mathfrak{H}_{S/R}$. We call $\mathfrak{H}_{S/R}$ the *homological different* of S over R.

The main object of § 3 is to show that if R is a noetherian integrally closed domain with field of quotients K, L a finite field extension of K and S an integral extension of R in L, then $\mathfrak{H}_{S/R}$ is contained in $\mathfrak{D}_{S/R}$, where $\mathfrak{D}_{S/R}$ is the usual different defined using the trace mapping on L. There are examples which show that $\mathfrak{H}_{S/R} \neq \mathfrak{D}_{S/R}$ in general. However, in the case that S is R-projective, we have that $\mathfrak{H}_{S/R} = \mathfrak{D}_{S/R}$. As an application we prove that if R is a regular local ring of dimension less than or equal to two and L is a separable field extension, then S is unramified over R if and only if each minimal prime ideal is unramified over R. This result has been obtained independently by J.-P. Serre (unpublished) and has been used by M. Nagata to prove the theorem in general i.e. for regular local rings of arbitrary dimension [7].

§ 4 is devoted to showing that under various conditions if S is unramified over R, then S is R-projective. For instance, this is the case if R is a noetherian, integrally closed domain and S is an integral extension of R in a finite, separable field extension of the field of quotients of R.

§ 5 is devoted to certain homological considerations and is the only place in the paper where any homology theory is used. Perhaps the most striking result obtained here is that if S is unramified over R and R-projective and T

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is a regular ring of finite Krull dimension (i.e. T is a noetherian ring and every local ring of T is a regular local ring) containing R, then $S \otimes_R T$ is regular if it is noetherian.

It has recently been brought to our attention that E. Noether had considered the ideals $\mathfrak{J}$, $\mathfrak{N}$ and $\mathfrak{H}_{S/R}$ from a somewhat different point of view in [5].

1. Notation and terminology. All rings to be considered in this paper will be assumed commutative with identity element and all modules will be unitary. A ring S together with a ring homomorphism $f: R \to S$ such that $f(1) = 1$ will be called an *R-algebra*. Ideals in S will be denoted by capital German letters and ideals in R by lower case German letters. If $\mathfrak{A}$ is an ideal in S, then we call the ideal $f^{-1}(\mathfrak{A})$ in R the *contraction of* $\mathfrak{A}$ and denote it by $\mathfrak{A} \cap R$. We shall consider $R/\mathfrak{A} \cap R$ to be a subring of $S/\mathfrak{A}$, the identification being given by the monomorphism induced by f. If $x \in R$, then we denote $f(x)$ in S by x, provided there is no danger of confusion. Thus if $\mathfrak{a}$ is an ideal in R, we shall denote the ideal $S \cdot f(\mathfrak{a})$ by $S \cdot \mathfrak{a}$. It is clear that $(S \cdot \mathfrak{a}) \cap R$ contains $\mathfrak{a}$.

A subset U of a ring is called a *multiplicative* system if a) whenever u_1 and u_2 are in U, the product $u_1 u_2$ is in U, and b) 0 is not in U; 1 is in U. The only multiplicative systems in R that we shall consider are those that do not meet the kernel of f. Suppose U is a multiplicative system in R and U' is a multiplicative system in S containing U (i.e. U' contains $f(U)$). Then $f: R \to S$ induces a homomorphism $f': R_U \to S_{U'}$ which makes $S_{U'}$ an R_U-algebra. In particular, if $\mathfrak{P}$ is a prime ideal in S and $\mathfrak{p} = \mathfrak{P} \cap R$, then $S_{\mathfrak{P}}$ is an $R_{\mathfrak{p}}$-algebra and $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is a subfield of $S_{\mathfrak{P}}/\mathfrak{P}S_{\mathfrak{P}}$.

We define the R-algebra S^e to be the ring $S \otimes_R S$ with the mapping of R into S^e being given by $r \to r \otimes 1$. S^e is called the enveloping algebra of the R-algebra S. The map $\phi: S^e \to S$ defined by $\phi(x \otimes y) = xy$ is an R-algebra epimorphism. Thus all S-modules may be considered as S^e-modules. The kernel of ϕ will be denoted by $\mathfrak{J}$ which is the ideal generated by $\{1 \otimes x - x \otimes 1\}$, and the annihilator of $\mathfrak{J}$ will be denoted by $\mathfrak{N}$. If U is a multiplicative system, we denote by $U \otimes U$ the multiplicative system in S^e consisting of all elements of the form $u_1 \otimes u_2$ where u_1, u_2 are in U. The map $(S_U)^e \to (S^e)_{U \otimes U}$ defined by $x/u_1 \otimes y/u_2 \to x \otimes y/u_1 \otimes u_2$ is an R-algebra isomorphism which we shall use to identify these algebras. If E is an S-module, then $E_U = E_{U \otimes U}$. Finally, if V is a multiplicative system in R such that V is contained in U, then $S_U \otimes_R S_U = S_U \otimes_{R_V} S_U$.

A prime ideal $\mathfrak{P}$ in the R-algebra S is said to be *unramified* if $\mathfrak{p} = R \cap \mathfrak{P}$ has the following properties:

a) $\mathfrak{p}S_{\mathfrak{P}} = \mathfrak{P}S_{\mathfrak{P}}$;

b) $S_{\mathfrak{P}}/\mathfrak{p}S_{\mathfrak{P}}$ is a separable field extension of $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$.[2]

The R-algebra S is said to be *unramified* if

a) every prime ideal in S is unramified;

b) for each prime ideal $\mathfrak{p}$ in R there are only a finite number of prime ideals $\mathfrak{P}$ in S such that $\mathfrak{p} = \mathfrak{P} \cap R$.

In other words, S is unramified if and only if given any prime ideal $\mathfrak{p}$ in R which is the contraction of a prime ideal in S and $U = R - \mathfrak{p}$, then $S_U/\mathfrak{p}S_U$ is a separable $R_U/\mathfrak{p}R_U$-algebra.

2. Ramification criteria.

LEMMA 2.1. *If S is an R-algebra, the following statements are equivalent:*

a) *S is S^e-projective;*

b) *the exact sequence $0 \to \mathfrak{J} \to S^e \xrightarrow{\phi} S \to 0$ splits;*

c) *there is an element z in S^e such that $z(1 \otimes x) = z(x \otimes 1)$ for all $x \in S$, and $\phi(z) = 1$;*

d) *$\phi(\mathfrak{N}) = S$.*

Further, if R is a field, then S is S^e-projective if and only if S is a separable R-algebra.

Proof. Clearly a) implies b). If b) holds, there is an S^e-homomorphism $p: S \to S^e$ such that ϕp is the identity. Let $z = p(1)$. It is easy to see that z is the desired element to make c) hold, so b) implies c).

Now suppose that c) is true. Then z is in $\mathfrak{N}$ so that $\phi(\mathfrak{N})$ contains $\phi(z) = 1$. Hence $\phi(\mathfrak{N}) = S$. Thus c) implies d).

If d) holds, there is a z in $\mathfrak{N}$ such that $\phi(z) = 1$. Define $p: S \to S^e$ by $p(s) = sz$. Since z is in $\mathfrak{N}$, p is an S^e-homomorphism so that b) holds. Clearly b) implies a) so that d) implies a).

The last statement in the theorem is found in [6, Theorem 1].

PROPOSITION 2.2. *If the R-algebra S is S^e-projective, then S is unramified.*

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[2] If R is a field, an R-algebra S is said to be *separable* if S is a finite-dimensional R-algebra which is a direct sum of separable field extensions of R.

Proof. We must show that if $\mathfrak{p}$ is any prime ideal in R where $\mathfrak{p} = \mathfrak{P} \cap R$ for some prime $\mathfrak{p} \subset S$ and $U = R - \mathfrak{p}$, then $S_U/\mathfrak{p}S_U$ is a separable $R_U/\mathfrak{p}R_U$-algebra. Since $R_U/\mathfrak{p}R_U$ is a field, we need only show that $S_U/\mathfrak{p}S_U$ is $S_U/\mathfrak{p}S_U \otimes S_U/\mathfrak{p}S_U$-projective, where the tensor product is taken over $R_U/\mathfrak{p}R_U$.

Since S is S^e-projective, there is a z in S^e such that $\phi(z) = 1$ and $z(x \otimes 1) = z(1 \otimes x)$ for all $x \in S$. We also have the commutative diagram

$$\begin{array}{ccccc} S \otimes_R S & \xrightarrow{f} & S_U \otimes_{R_U} S_U & \xrightarrow{g} & S_U/\mathfrak{p}S_U \otimes_{R_U} S_U/\mathfrak{p}S_U \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi''} \\ S & \longrightarrow & S_U & \longrightarrow & S_U/\mathfrak{p}S_U \end{array}$$

where the maps are the obvious ones. Let $z'' = gf(z)$. Then it is clear that $\phi''(z'') = 1$ and that z'' is in the annihilator of the kernel of ϕ''. But since $S_U/\mathfrak{p}S_U \otimes_{R_U} S_U/\mathfrak{p}S_U = S_U/\mathfrak{p}S_U \otimes_F S_U/\mathfrak{p}S_U$ (where $F = R_U/\mathfrak{p}R_U$) we have by 2.1 that S is unramified.

PROPOSITION 2.3. *Let R be an integral domain with field of quotients K. If S is an R-algebra such that S is S^e-projective, then $S \otimes_R K$ is a separable K-algebra. In particular, if S is an integral domain, then its field of quotients is a separable field extension of K.*

Proof. It is easy to see that condition c) of 2.1 holds for the K-algebra $S \otimes_R K$ since it holds for S. If S is an integral domain, then $S \otimes_R K$ is also and is therefore the field of quotients of S. Hence the second part of the proposition is true.

PROPOSITION 2.4. *Let S be a noetherian ring which is an R-algebra such that every maximal ideal in S is unramified. Then any R-derivation D of S into a finitely generated S-module E is zero.*

Proof. Let $\mathfrak{M}$ be a maximal ideal of S and $\mathfrak{m} = \mathfrak{M} \cap R$. Denote by $E_{\mathfrak{M}}$ the $S_{\mathfrak{M}}$-module $E \otimes_S S_{\mathfrak{M}}$ and by $D_{\mathfrak{M}}\colon S_{\mathfrak{M}} \to E_{\mathfrak{M}}$ the derivation induced by $D\colon S \to E$.

Since $\mathfrak{M}$ is unramified, we have $\mathfrak{m}S_{\mathfrak{M}} = \mathfrak{M}S_{\mathfrak{M}}$. Since D is a derivation over R, $D_{\mathfrak{M}}$ is a derivation over $R_{\mathfrak{m}}$ so that $D_{\mathfrak{M}}(\mathfrak{m}S_{\mathfrak{M}})$ is contained in $\mathfrak{m}E_{\mathfrak{M}} = \mathfrak{M}E_{\mathfrak{M}}$. Therefore $D_{\mathfrak{M}}$ induces a derivation

$$\bar{D}_{\mathfrak{M}}\colon S_{\mathfrak{M}}/\mathfrak{M}S_{\mathfrak{M}} \to E_{\mathfrak{M}}/\mathfrak{M}E_{\mathfrak{M}}$$

over $R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}$. But since $\mathfrak{M}$ is unramified, $S_{\mathfrak{M}}/\mathfrak{M}S_{\mathfrak{M}}$ is a separable extension of $R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}$ so that $\bar{D}_{\mathfrak{M}} = 0$. Hence $D_{\mathfrak{M}}(S_{\mathfrak{M}})$ is contained in $\mathfrak{M}E_{\mathfrak{M}}$.

Iterating this argument, we have that $D_{\mathfrak{M}}(S_{\mathfrak{M}})$ is contained in $\cap \mathfrak{M}^i E_{\mathfrak{M}} = 0$ so that $D_{\mathfrak{M}} = 0$.

Since $D_{\mathfrak{M}} = 0$ for every maximal ideal $\mathfrak{M}$ of S, it follows easily that $D = 0$.

THEOREM 2.5. *Let S be a noetherian R-algebra such that $\mathfrak{J}$ is a finitely generated ideal in S^e. Then the following statements are equivalent:*

a) *S is S^e-projective;*

b) *S is unramified;*

c) *Every maximal ideal in S is unramified;*

d) *Every R-derivation of S into a finitely generated S-module is zero.*

Proof. We have already shown in 2.2 that a) implies b) and clearly b) implies c). Proposition 2.4 shows that c) implies d). Hence we need only prove that d) implies a).

To show that S is S^e-projective, it suffices to show that the exact sequence

$$0 \to \mathfrak{J} \to S^e \to S \to 0 \tag{1}$$

splits.

Observe first that for any S-module E, $\mathrm{Hom}_S(\mathfrak{J}/\mathfrak{J}^2, E)$ is isomorphic to the group of R-derivations of S into E. Letting $E = \mathfrak{J}/\mathfrak{J}^2 = S \otimes_{S^e} \mathfrak{J}$, and observing that $\mathfrak{J}/\mathfrak{J}^2$ is a finitely generated S-module (since $\mathfrak{J}$ is assumed to be finitely S^e-generated), we have $\mathrm{Hom}_S(\mathfrak{J}/\mathfrak{J}^2, \mathfrak{J}/\mathfrak{J}^2) = 0$ and hence $\mathfrak{J}/\mathfrak{J}^2 = 0$, i.e. $\mathfrak{J} = \mathfrak{J}^2$. Since $\mathfrak{J}$ is finitely generated, there is a y_0 in $\mathfrak{J}$ such that $x = y_0 x$ for each $x \in \mathfrak{J}$.

Now define a map $p: S^e \to \mathfrak{J}$ by letting $p(1) = y_0$. Then for all $x \in \mathfrak{J}$, we have $p(x) = xp(1) = xy_0 = x$. Thus the sequence (1) above splits and we are done.

LEMMA 2.6. *Let S be a ring, E a finitely generated S-module, and $\mathfrak{A}$ the annihilator of E. If U is a multiplicative system in S, then the annihilator in S_U of E_U is $\mathfrak{A}_U$.*

Proof. It is obvious that $\mathfrak{A}_U$ is contained in the annihilator of E_U. Suppose, then, that $(s/u)E_U = 0$. Let E be generated by $e_1, \cdots, e_n$. Then E_U is also generated by $e_1, \cdots, e_n$ and we have $(s/u)e_i = 0$ for $i = 1, \cdots, n$. But then there is a u' in U such that $u'se_i = 0$ for $i = 1, \cdots, n$ so that $u's$ is in $\mathfrak{A}$. Hence s is in $\mathfrak{A}_U$ and so is s/u.

We now define the *homological different* of the R-algebra S to be the ideal $\phi(\mathfrak{N})$ in S, and denote it by $\mathfrak{H}_{S/R}$.

THEOREM 2.7. *Let S be a noetherian R-algebra such that $\mathfrak{J}$ in S^e is a finitely generated ideal. A prime ideal $\mathfrak{P}$ in S is unramified if and only if $\mathfrak{P}$ does not contain $\mathfrak{H}_{S/R}$.*

Proof. Let $\mathfrak{P}$ be a prime ideal in S, $\mathfrak{p} = \mathfrak{P} \cap R$, $U = S - \mathfrak{P}$, $V = R - \mathfrak{p}$. Then S_U is an R_V-algebra, $(S^e)_{U\otimes U} = S_U \otimes_R S_U = S_U \otimes_{R_V} S_U$, the kernel of map $\bar{\phi}: S^e{}_U \to S_U$ is $\mathfrak{J}_{U\otimes U}$, and $\mathfrak{N}_{U\otimes U}$ is the annihilator of $\mathfrak{J}_{U\otimes U}$. Moreover, S_U is a noetherian R_V-algebra and $\mathfrak{J}_{U\otimes U}$ is a finitely generated ideal in $(S_U)^e$.

Suppose $\mathfrak{P}$ does not contain $\mathfrak{H}_{S/R}$. Then $(\mathfrak{H}_{S/R})_U = S_U = \bar{\phi}(\mathfrak{N}_{U\otimes U})$ and so, by 2.1, S_U is $(S_U)^e$-projective. But then, by 2.5, S_U is an unramified R -algebra. It is thus easy to see that $\mathfrak{P}$ is unramified.

Now suppose that $\mathfrak{P}$ is unramified. Then $\mathfrak{P}S_U$ is unramified over R_V and again by 2.5 (since $\mathfrak{P}S_U$ is the only maximal ideal in S_U), S_U is $(S_U)^e$-projective. Thus by 2.1, $\bar{\phi}(\mathfrak{N}_{U\otimes U}) = S_U$. Since $\bar{\phi}(\mathfrak{N}_{U\otimes U}) = (\mathfrak{H}_{S/R})_U$, we have $(\mathfrak{H}_{S/R})_U = S_U$ which implies that $\mathfrak{P}$ does not contain $\mathfrak{H}_{S/R}$.

COROLLARY 2.8. *Let R be an integral domain with field of quotients K. If S is a noetherian R-algebra such that $\mathfrak{J}$ is a finitely generated ideal in S^e, and $S \otimes_R K$ is not a separable K-algebra, then every prime ideal in S is ramified.*

LEMMA 2.9. *If S is a noetherian ring, and E is a finitely generated S-module, then $\bigcap_{\mathfrak{M}} (\bigcap_i \mathfrak{M}^i E) = 0$, where $\mathfrak{M}$ runs through all maximal ideals of S.*

Proof. Let e be an element of $\bigcap_{\mathfrak{M}} (\bigcap_i \mathfrak{M}^i E)$, and let $\mathfrak{A}$ be the annihilator of e. If $e \neq 0$, then $\mathfrak{A}$ is a proper ideal of S and so is contained in some maximal ideal $\mathfrak{M}$. Since e is in $\bigcap \mathfrak{M}^i E$, there is an m in $\mathfrak{M}$ such that $(1-m)e = 0$. However, $(1-m)$ is not in $\mathfrak{A}$ so that e must be zero.

PROPOSITION 2.10. *Let S be a noetherian R-algebra, and $\mathfrak{J}$ finitely generated in S^e. Then S is unramified if and only if for every maximal ideal $\mathfrak{M}$ of S every R-derivation $D: S \to S/\mathfrak{M}$ is zero.*

Proof. Proposition 2.4 shows us that if S is unramified then every R-derivation $D: S \to S/\mathfrak{M}$ is zero. To prove the converse, it is sufficient (by 2.5) to show that every R-derivation $D: S \to E$ is zero, where E is any finitely generated S-module.

If $D: S \to E$ is an R-derivation, E finitely generated, and $\mathfrak{M}$ a maximal ideal of S, then $E/\mathfrak{M}E$ is a finite-dimensional vector space over $S/\mathfrak{M}$ and so the derivation $\bar{D}: S \to E/\mathfrak{M}E$ is zero, where $\bar{D}$ is the composition

$S \xrightarrow{D} E \to E/\mathfrak{M}E$. Therefore $D(S)$ is contained in $\mathfrak{M}E$. Iterating this (i.e. observing that $\mathfrak{M}^iE/\mathfrak{M}^{i+1}E$ is a finite-dimensional vector space over $S/\mathfrak{M}$) we see that $D(S)$ is contained in $\bigcap \mathfrak{M}^iE$. Since this is so for all maximal ideals $\mathfrak{M}$ of S, we have $D(S)$ is contained in $\bigcap_{\mathfrak{M}} (\bigcap_i \mathfrak{M}^iE) = 0$ (by 2.9). Hence $D = 0$.

PROPOSITION 2.11. *Let S and T be R-algebras such that $\mathfrak{J} = \operatorname{Ker}(S^e \xrightarrow{\phi} S)$ is finitely generated in S^e and S is noetherian. If S is unramified, then $S \otimes_R T$ is unramified as a T-algebra. If, further, T is unramified, then $S \otimes_R T$ is unramified as an R-algebra.*

Proof. Since S is unramified, we have by 2.5 that the exact sequence

$$0 \to \mathfrak{J} \to S^e \to S \to 0$$

splits. Therefore the sequence

$$0 \to \mathfrak{J} \otimes_R T \to S^e \otimes_R T \to S \otimes_R T \to 0$$

is exact and splits. But $S^e \otimes_R T = (S \otimes_R T) \otimes_T (S \otimes_R T) = (S \otimes_R T)^e$ as a T-algebra. Hence $S \otimes_R T$ is $(S \otimes_R T)^e$-projective. Thus, by 2.2, it follows that $S \otimes_T T$ is an unramified T-algebra.

If, in addition, T is an unramified R-algebra, then it easily follows from the fact that $S \otimes_R T$ is an unramified T-algebra, that $S \otimes_R T$ is an unramified R-algebra.

3. The homological different and different. Throughout this section, R will be an integral domain with field of quotients K, L a finite-dimensional K-algebra, and S a subring of L containing R such that $S \otimes_R K = L$. We shall denote $\operatorname{Hom}_R(S, R)$ by S^* and $\operatorname{Hom}_K(L, K)$ by L^*.

We define the map $\tau: S \otimes_R S \to \operatorname{Hom}_R(S^*, S)$ to be $\tau(x \otimes y)(f) = xf(y)$ for f in S^*. By [3; VI, 5.2] if S is a projective and finitely generated R-module, then τ is an isomorphism. In particular, $\sigma: L \otimes_K L \to \operatorname{Hom}_K(L^*, L)$ which is similarly defined, is an isomorphism.

Since $S \otimes_R K = L$, every element of S^* is uniquely extendable to an element of L^*. If S^* generates all of L^* over K, then we have a natural map $\rho: \operatorname{Hom}_R(S^*, S) \to \operatorname{Hom}_K(L^*, L)$. We therefore obtain the following diagram which is easily shown to be commutative:

$$\begin{array}{ccccccccc} & & & & \mathrm{Hom}_S(S^*,S) & \longrightarrow & \mathrm{Hom}_L(L^*,L) & & \\ & & & & \downarrow & & \downarrow & & \\ S\otimes_R S & \xrightarrow{\tau} & \mathrm{Hom}_R(S^*,S) & \xrightarrow{\rho} & \mathrm{Hom}_K(L^*,L) & \xrightarrow{\sigma^{-1}} & L\otimes_K L & & \\ \downarrow{\scriptstyle\phi} & & & & & & & \downarrow{\scriptstyle\phi'} & \\ S & \longrightarrow & & & & & & & L. \end{array}$$

If f is in $\mathrm{Hom}_L(L^*,L)$, it can be seen by standard techniques of linear algebra that $\phi'\sigma^{-1}(f)=f(\mathrm{Tr})$, where $\mathrm{Tr}\colon L\to K$ is the trace map.

An explicit description of ρ and of $\phi'\sigma^{-1}\rho$ can be given as follows: let $v^1,\cdots,v^n$ be elements of S^* which form a basis for L^* over K. Then for f in $\mathrm{Hom}_R(S^*,S)$ we have $\rho(f)$ defined by $\rho(f)(v^i)=f(v^i)$. If we let $v_1,\cdots,v_n$ be the basis of L over K dual to $v^1,\cdots,v^n$ and set $v_i=u_i/r_0$ with u_i in S, r_0 in R (using the fact that $S\otimes_R K=L$), we can easily see that every element s of S can be written $s=\sum r_iv_i$ with r_i in R and that $\mathrm{Tr}(r_0S)$ is contained in R. Thus if we let $T'\colon S\to R$ be the restriction of $r_0\mathrm{Tr}$ to S, we have for f in $\mathrm{Hom}_S(S^*,S)$ that $\phi'\sigma^{-1}\rho(f)=(1/r_0)f(T')$. If $\mathrm{Tr}(S)$ were contained in R (e.g. if S were integral over R) then $\phi'\sigma^{-1}\rho(f)=f(\mathrm{Tr}')$ where Tr' is the restriction of Tr to S.

We now define the *complementary module*, $\mathfrak{C}_{S/R}$, and the *different*, $\mathfrak{D}_{S/R}$, as follows:

$$\mathfrak{C}_{S/R}=\{x \text{ in } L/\mathrm{Tr}(xS) \text{ is contained in } R\}$$

$$\mathfrak{D}_{S/R}=\{x \text{ in } L/x\mathfrak{C}_{S/R} \text{ is contained in } S\}.$$

From the above remarks, we can see that $\phi'\sigma^{-1}\rho(\mathrm{Hom}_S(S^*,S))$ is contained in $\mathfrak{D}_{S/R}$. For suppose x is in $\mathfrak{C}_{S/R}$ and f is in $\mathrm{Hom}_S(S^*,S)$. Then $x[(1/r_0)f(T')]=x(\rho(f)(\mathrm{Tr}))=\rho(f)(\mathrm{Tr}\circ x)$, where $\mathrm{Tr}\circ x\cdot L\to K$ is defined by $\mathrm{Tr}\circ x(y)=\mathrm{Tr}(xy)$ for y in L. Since $\mathrm{Tr}\circ x$ restricted to S maps S into R, $\rho(f)(\mathrm{Tr}\circ x)$ is in S. Therefore $x\cdot\phi'\sigma^{-1}\rho(f)$ is in S for all x in $\mathfrak{C}_{S/R}$.

Now let us make $\mathrm{Hom}_R(S^*,S)$ an S^e-module by defining $(x\otimes y)f(g)=x\cdot f(g\circ y)$ for x, y in S, f in $\mathrm{Hom}_R(S^*,S)$, g in S^*. Then τ is an S^e-homomorphism. Furthermore, $\mathrm{Hom}_S(S^*,S)$ is equal to the set of all f in $\mathrm{Hom}_R(S^*,S)$ such that $\mathfrak{J}f=0$. Thus, since $\mathfrak{N}=$ annihilator of $\mathfrak{J}$ in S^e, we have $\tau(\mathfrak{N})$ is contained in $\mathrm{Hom}_S(S^*,S)$.

We can go even a little further. Let

$$\mathfrak{W}=\mathrm{Ker}(S\otimes_R S\to L\otimes_K L=S\otimes_R S\otimes_R K),$$

and let $\mathfrak{A}$ in S^e be the annihilator of $\mathfrak{J}/\mathfrak{W}$ ($\mathfrak{W}$ is obviously contained in $\mathfrak{J}$).

Since $\mathfrak{W}$ is the torsion submodule of S^e (as an R-module), and since S is torsion-free as an R-module, $\tau(w) = 0$ for all w in $\mathfrak{W}$. Thus, if a is in $\mathfrak{A}$, we have $a\mathfrak{J}$ is contained in $\mathfrak{W}$ and $0 = \tau(a\mathfrak{J}) = \tau(a)\mathfrak{J}$. Thus by the remark above $\tau(a)$ is in $\mathrm{Hom}_S(S^*, S)$ which implies that $\tau(\mathfrak{A})$ is contained in $\mathrm{Hom}_S(S^*, S)$.

Combining all the above remarks, and resorting to the commutative diagram above, we have shown

PROPOSITION 3.1. *Let R be an integral domain with field of quotients K, L a finite-dimensional K-algebra, and S a subring of L containing R such that $S \otimes_R K = L$ and $S^* \otimes_R K = L^*$. Then if $\mathfrak{A}$ is the annihilator of $\mathfrak{J}/\mathfrak{W}$, $\phi(\mathfrak{A})$ is contained in $\mathfrak{D}_{S/R}$. In particular, $\mathfrak{H}_{S/R}$ is contained in $\mathfrak{D}_{S/R}$.*

PROPOSITION 3.2. *Let R, K, L and S be as in* 3.1 *and in addition assume L is a field. Then L is a separable extension of K if and only if $\mathfrak{H}_{S/R} \neq 0$ (i.e. $\mathfrak{N}$ is not contained in $\mathfrak{J}$).*

Proof. If L is not separable, the trace map is identically zero, so that for all f in $\mathrm{Hom}_S(S^*, S)$, $\rho(f)(\mathrm{Tr}) = 0$. Since $\mathfrak{H}_{S/R} = \phi(\mathfrak{N}) = \phi'\sigma^{-1}\rho\tau(\mathfrak{N})$ and $\tau(\mathfrak{N})$ is contained in $\mathrm{Hom}_S(S^*, S)$, we have $\mathfrak{H}_{S/R} = 0$.

If L is separable, the exact sequence

$$0 \to \mathfrak{J}' \to L \otimes_K L \to L \to 0$$

splits. However, $L = S \otimes_R K$, $L \otimes_K L = S^e \otimes_R K$, and $\mathfrak{J}' = \mathfrak{J} \otimes_R K$. Thus the annihilator $\mathfrak{N}'$ of $\mathfrak{J}'$ is $\mathfrak{N} \otimes_R K$ (by 2.6), and $\mathfrak{N}'$ is not contained in $\mathfrak{J}'$. Therefore $\mathfrak{N}$ is not contained in $\mathfrak{J}$ and $\mathfrak{H}_{S/R} \neq 0$.

PROPOSITION 3.3. *Let R be an integrally closed integral domain with field of quotients K, L a separable K-algebra, and S a subring of L containing R which is integral over R and such that $S \otimes_R K = L$. Then $\mathrm{Hom}_S(S^*, S)$ is isomorphic to $\mathfrak{D}_{S/R}$ under the map $f \to f(\mathrm{Tr})$. If S is a projective, finitely generated R-module, then $\mathfrak{H}_{S/R} = \mathfrak{D}_{S/R}$.*

Proof. Since L is a separable K-algebra, the map $L \to L^*$ given by $x \to \mathrm{Tr} \circ x$ is an isomorphism. Under this isomorphism, $\mathfrak{C}_{S/R}$ is mapped onto S^*. Thus $\mathrm{Hom}_S(S^*, S) \approx \mathrm{Hom}_S(\mathfrak{C}_{S/R}, S)$ and this latter module is isomorphic to $\mathfrak{D}_{S/R}$. The composite isomorphism is the one described above, namely $f \to f(\mathrm{Tr})$ (Tr is here restricted to S). Furthermore, by standard arguments, it is easy to see that $S^* \otimes_R K = L^*$ so that all our previous discussion (including commutative diagram) holds. Moreover, if S is finitely generated over R and R-projective, then τ is an isomorphism, $\mathfrak{W} = 0$, and $\tau(\mathfrak{N}) = \mathrm{Hom}_S(S^*, S)$. Thus in this case $\mathfrak{H}_{S/R} = \mathfrak{D}_{S/R}$.

Throughout the rest of this section we shall denote the R-module $\mathrm{Hom}_R(E,R)$ by E^*, where E is an arbitrary R-module.

PROPOSITION 3.4. *Let R be a noetherian domain such that every proper principal ideal is unmixed (e.g. R is integrally closed). Let A be a finitely generated R-module such that $A = A^{**}$, and B a finitely generated torsion-free R-module containing A such that B/A is a non-trivial torsion module. Then $\mathfrak{a}(B/A)$ is unmixed of rank one ($\mathfrak{a}(B/A)$ is the annihilator of B/A in R).*[3]

Proof. Let $b_1, \cdots, b_t$ be generators of B. Then $\mathfrak{a}(B/A) = \{r \text{ in } R/rb_i \text{ is in } A \text{ for } i = 1, \cdots, t\} = \cap\, \mathfrak{a}_i$, where $\mathfrak{a}_i = \{r \text{ in } R/rb_i \text{ is in } A\}$. Thus if each $\mathfrak{a}_i$ is unmixed of rank one, then so is $\mathfrak{a}(B/A)$. We may therefore suppose that $B = A + Rb$, b not in A, and B/A is a torsion module.

Observe next that if h is in A^*, then h can be extended to a map $\bar{h}: A \otimes_R K \to K$. Moreover, if x in $A \otimes_R K$ is such that $\bar{h}(x) = 0$ for all h in A^*, then $x = 0$. As a result, we have that r is in $\mathfrak{a}(B/A)$ if and only if $r\bar{h}(b) = \bar{h}(rb)$ is in R for all h in A^*. For if r is in $\mathfrak{a}(B/A)$, then rb is in A so that $\bar{h}(rb) = h(rb)$ is in R. Conversely, if $\bar{h}(rb)$ is in R for all h in A^*, then the map $A^* \to R$ given by $h \to \bar{h}(rb)$ is an element of $A^{**} = A$ so that $\bar{h}(rb) = h(a_0)$ for some a_0 in A and all h in A^*. Thus $\bar{h}(rb - a_0) = 0$ for all h in A^* and by the above remarks, $rb = a_0$ i.e. r is in $\mathfrak{a}(B/A)$.

Since A^* is finitely generated, say by $h_1, \cdots, h_n$, we see that $\mathfrak{a}(B/A) = \{r \text{ in } R/\bar{h}_i(rb) \text{ is in } R \text{ for } i = 1, \cdots, n\}$. Let $\bar{h}_i(b) = u_i/v$. Then r is in $\mathfrak{a}(B/A)$ if and only if r is in $\cap\,(v) : u_i$ i.e. $\mathfrak{a}(B/A) = \cap\,(v) : u_i$. Now by assumption on R, (v) is an unmixed ideal of rank one so that $(v) : u_i$ is also. Thus $\mathfrak{a}(B/A)$ is unmixed of rank one.

COROLLARY 3.5. *Let R be an integrally closed noetherian integral domain with field of quotients K, L a separable field extension of K, and S the integral closure of R in L. Then if $\mathfrak{D}_{S/R} \neq S$, $\mathfrak{D}_{S/R}$ must be of rank one.*

This follows from 3.4 by letting $R = A = S$ and $B = \mathfrak{C}_{S/R}$ and observing that $\mathfrak{D}_{S/R} = \mathfrak{a}(\mathfrak{C}_{S/R}/S)$.

PROPOSITION 3.6. *With R, K, L and S as above, we have $\mathfrak{D}_{S/R} = S$ if and only if every minimal prime ideal of S is unramified.*

Proof. By 2.7, every minimal prime ideal of S is unramified if and only if rank $\mathfrak{H}_{S/R}$ is greater than one. Since $\mathfrak{D}_{S/R}$ contains $\mathfrak{H}_{S/R}$, we have that if every minimal prime is unramified, then $\mathfrak{D}_{S/R}$ has rank greater than one. Thus, by 3.5, $\mathfrak{D}_{S/R} = S$.

[3] We would like to thank O. Goldman for suggesting this proposition to us.

Conversely, let $\mathfrak{D}_{S/R} = S$, $\mathfrak{P}$ be a minimal prime of S, and $\mathfrak{p} = \mathfrak{P} \cap R$. Then $\mathfrak{p}$ is a minimal prime of R and $R_\mathfrak{p}$ is a regular local ring of dimension one. Since $S_\mathfrak{p}$ is a finitely generated, torsion-free $R_\mathfrak{p}$-module, $S_\mathfrak{p}$ is $R_\mathfrak{p}$-free. We have by 3.3 that $\mathfrak{H}_{S_\mathfrak{p}/R_\mathfrak{p}} = \mathfrak{D}_{S_\mathfrak{p}/R_\mathfrak{p}}$. But it is easily seen that $\mathfrak{H}_{S_\mathfrak{p}/R_\mathfrak{p}} = \mathfrak{H}_{S/R} \otimes_S S_\mathfrak{p}$ and $\mathfrak{D}_{S_\mathfrak{p}/R_\mathfrak{p}} = \mathfrak{D}_{S/R} \otimes_S S_\mathfrak{p}$. Therefore $\mathfrak{H}_{S_\mathfrak{p}/R_\mathfrak{p}} = S_\mathfrak{p}$ and thus $\mathfrak{P}$ is unramified.

COROLLARY 3.7. *Let R, K, S and L be as in 3.5 and assume further that S is R-projective. Then S is unramified if and only if every minimal prime ideal of S is unramified.*

Proof. Since S is R-projective and finitely generated, we have $\mathfrak{H}_{S/R} = \mathfrak{D}_{S/R}$. Furthermore, S is unramified if and only if $\mathfrak{H}_{S/R} = S$ (by 2.5). Thus 3.6 implies 3.7.

The following theorem has also been obtained independently by Serre.

THEOREM 3.8. *Let R be a regular local ring of dimension less than or equal to two, and let K, S, and L be as above. Then S is unramified if and only if every minimal prime ideal of S is unramified.*

Proof. We need only show that S is R-projective, for then we may apply 3.7. However, by [2, 2.10] it is sufficient to show that S is a Macaulay ring (S is semi-local). Since $\dim S \leqq 2$, and since S is integrally closed, we have that every principal ideal of S is unmixed and that every ideal of rank two that is generated by two elements is unmixed. Hence S is Macaulay and therefore R-projective.

4. On being free. Throughout this section, R will be an integrally closed local domain with maximal ideal $\mathfrak{m}$ and field of quotients K. The residue class field $R/\mathfrak{m}$ will be denoted by F.

PROPOSITION 4.1. *Let L be a separable K-algebra and S an integral extension of R in L such that $S \otimes_R K = L$ and is unramified. Then $\mathfrak{D}_{S/R} = S$.*

Proof. Since S is unramified, $\mathfrak{H}_{S/R} = S$. However, since $\mathfrak{H}_{S/R}$ is contained in $\mathfrak{D}_{S/R}$, we have $\mathfrak{D}_{S/R} = S$.

LEMMA 4.2. *Let S be an R-algebra containing R which is torsion-free over R and such that*

a) *S is finitely generated over R,*

b) *there is an element t in $S/\mathfrak{m}S$ such that $S/\mathfrak{m}S = F[t]$.*

If θ in S is such that $\theta \to t$ under the natural map $S \to S/\mathfrak{m}S$, then

$S = R[\theta]$ and $\{1, \theta, \cdots, \theta^{n-1}\}$ is a free basis for S over R (where $n = [S/\mathfrak{m}S : F]$).

Proof. Since $S/\mathfrak{m}S = F[t]$, we have that $\{1, t, \cdots, t^{n-1}\}$ is a basis for $S/\mathfrak{m}S$ over F. Since R is a local ring, this implies that $\{1, \theta, \cdots, \theta^{n-1}\}$ generates S over R and is a minimal generating set for S over R. Also, since R is integrally closed, we know that the minimal polynomial f in $K[x]$ for θ has its coefficients in R. We will show that degree $f = n$, hence that $\{1, \theta, \cdots, \theta^{n-1}\}$ is a basis for $K[\theta]$ over K. This will imply that $\{1, \theta, \cdots, \theta^{n-1}\}$ is a free basis for S over R.

Let $\bar{f}$ in $F[X]$ be the corresponding polynomial of f. Then $\bar{f}(t) = 0$ so that $\deg \bar{f} \geqq n$. On the other hand, since θ^n is in S, we have $\theta^n = \sum_{i=0}^{n-1} r_i \theta^i$, r_i in R. Therefore $\deg f \leqq n$ and we are done.

PROPOSITION 4.3. *Let S be a torsion-free R-algebra containing R which is unramified and finitely generated over R. Then S is a free R-module on n generators (where $n = [S/\mathfrak{m}S : F]$). Moreover, if L is the full ring of quotients of S, $[L : K] = n$ and S is integrally closed in L.*

Proof. Let us assume first that F is an infinite field. Then it is well known [4] that $S/\mathfrak{m}S = F[t]$. Thus, by 4.2, we have that S is R-free with basis $\{1, \theta, \cdots, \theta^{n-1}\}$.

Now suppose F is finite. Let X be an indeterminate, and consider the local domain $R[X]_{\mathfrak{m}^*} = R'$, where $\mathfrak{m}^*$ is the extension of $\mathfrak{m}$ to $R[X]$. The maximal ideal $\mathfrak{m}'$ of R' is $R'\mathfrak{m}^*$ and $R'/\mathfrak{m}' = F' = F(X)$. R' is integrally closed since $R[X]$ is and rings of quotients of integrally closed rings are integrally closed.

We now have R' contained in S', where $S' = S[X]_{\mathfrak{m}^*}$, S' a finitely generated torsion-free R'-module, and $[S'/\mathfrak{m}'S' : F'] = n$. If we show that S' is unramified over R', and use the fact that F' is infinite, we will have that S' is a free R'-module on n generators. This will imply that S is R-free on n generators for if $s_1, \cdots, s_m$ is a minimal generating set for S over R, it is also one for S' over R', hence a free basis for S' over R' and therefore a free basis for S over R, with $m = n$.

Since S is unramified, S is S^e-projective so that the exact sequence

$$0 \to \mathfrak{J} \to S^e \to S \to 0$$

splits. Therefore, the exact sequence

$$0 \to \mathfrak{J} \otimes_R R' \to S^e \otimes_R R' \to S \otimes_R R' \to 0$$

splits. Since $S' = S[X] \otimes_{R[X]} R' = S \otimes_R (R[X] \otimes_{R[X]} R') = S \otimes_R R'$ and $(S')^e = S' \otimes_{R'} S' = S^e \otimes_R R'$, we see that S' is $(S')^e$-projective and therefore S' is unramified over R'. This then shows that S is R-free on n generators. The rest of the proposition follows from standard arguments [4].

THEOREM 4.4. *Let R be a noetherian integral domain (not necessarily local) with field of quotients K, and L a separable K-algebra. If S is an unramified integral extension of R in L such that $S \otimes_R K = L$, then S is R-projective.*

Proof. By standard localization arguments, this result follows from 4.3.

PROPOSITION 4.5. *Let S be a local ring containing R which is unramified and finitely generated over R. Then S is R-free.*

Proof. Since S is unramified, $S \otimes_R K$ is a separable K-algebra. Let $S' = \mathrm{Im}(S \to S \otimes_R K)$. Then S' is torsion-free and finitely generated over R, and we have the exact sequence

$$0 \to t(S) \to S \to S' \to 0, \tag{E}$$

where $t(S)$ is the R-torsion submodule of S, and is finitely generated over R. If we can show that $t(S)/\mathfrak{m}t(S) = 0$, we will have $t(S) = 0$. Therefore $S \approx S'$ and so S will be torsion-free, hence free (by 4.3). Since S is unramified, $S/\mathfrak{m}S$ is a field and therefore the map $S/\mathfrak{m}S \to S'/\mathfrak{m}S'$, being an epimorphism, must be an isomorphism. Moreover, $S/\mathfrak{m}S$ is a separable extension of F so that $S'/\mathfrak{m}S'$ is also. Hence S' is unramified (by 2.5 it is sufficient to test ramification of S' by its unique maximal ideal) and by 4.3 is free over R. Therefore the sequence (E) splits over R so that the sequence

$$0 \to t(S)/\mathfrak{m}t(S) \to S/\mathfrak{m}S \to S'/\mathfrak{m}S' \to 0$$

is exact. Since $S/\mathfrak{m}S \approx S'/\mathfrak{m}S'$, we have $t(S)/\mathfrak{m}t(S) = 0$, hence $t(S) = 0$ and $S \approx S'$.

PROPOSITION 4.6. *Let R be analytically normal (i.e. $\hat{R}$, the completion of R, is also an integrally closed local domain) and let S be a ring containing R which is unramified and finitely generated over R. Then S is R-free.*

Proof. $S\mathfrak{m}$ is the radical of S, so that $\hat{S}$ contains $\hat{R}$ and $\hat{S}$ is a finitely generated $\hat{R}$-module. Now $\hat{S} = S_1 + \cdots + S_n$ (direct sum), where each S_i is a local ring which is an $\hat{R}$-algebra. In fact, each S_i contains a copy of $\hat{R}$. It can also be easily seen that each S_i is an unramified $\hat{R}$-algebra (since S is unramified over R). Therefore, by 4.5, each S_i is free over $\hat{R}$, which implies

that $\hat{S}$ is R-free. Since S is finitely generated over R, $\hat{S}$ being R-free implies that S is R-free [1, Theorem 3.2].

Since an integrally closed geometric local ring is analytically normal, we see that an unramified integral, finitely generated extension ring S of an integrally closed geometric local ring R is R-free. Hence if R is a normal affine ring (not necessarily local), S is R-projective.

LEMMA 4.7. *Let R be a noetherian ring (not necessarily an integrally closed local domain) and let S be a ring containing R which is finitely generated as an R-module. If S is R-projective, then R is a direct summand of S as an R-module.*

Proof. From the exact sequence

$$0 \to R \to S \to S/R \to 0$$

it is clearly sufficient to prove that S/R is R-projective. Let $\mathfrak{m}$ be a maximal ideal of R. Then the exact sequence

$$0 \to R_{\mathfrak{m}} \to S_{\mathfrak{m}} \to (S/R)_{\mathfrak{m}} \to 0$$

splits since $S_{\mathfrak{m}}$ is a projective (hence free) $R_{\mathfrak{m}}$-module, and 1 is part of a free basis for $S_{\mathfrak{m}}$ over $R_{\mathfrak{m}}$. Therefore $(S/R)_{\mathfrak{m}}$ is free for every maximal ideal $\mathfrak{m}$ and so by [3, VII, Exercise 11] S/R is R-projective.

PROPOSITION 4.8. *Let $R \subset S \subset T$ be noetherian rings with T a finitely generated projective unramified R-algebra. Then S is unramified over R if and only if T is S-projective.*

Proof. Suppose T is S-projective. Then we have the commutative diagram

$$\begin{array}{ccc} S \otimes_R S & \longrightarrow & T \otimes_R T \\ \downarrow & & \downarrow \\ S & \longrightarrow & T. \end{array}$$

Since T is S-projective, $T \otimes_R T$ is $S \otimes_R S$-projective, and S is a direct summand of T as an S- hence also as an $S \otimes_R S$-module. But T is $T \otimes_R T$-projective since T is an unramified R-algebra. Hence S is $S \otimes_R S$-projective.

By [3, IX, Proposition 2.3] (letting $\Lambda = \Gamma = S$, $A = S$, $\Sigma = R$, $B = T$) we have that since S is $S \otimes_R S$-projective (being unramified) and T is R-projective, then T is S-projective.

5. Ramification and homology.

PROPOSITION 5.1. *Let S and T be R-algebras such that S is R-projective and S is S^e-projective. If E is an $S \otimes_R T$-module, then* $\mathrm{hd}_{S\otimes_R T} E = \mathrm{hd}_T E$ *and thus* $\mathrm{gl.dim}\, S \otimes_R T \leqq \mathrm{gl.dim}\, T$.

Further, if S is R-free, then $\mathrm{gl.dim}\, S \otimes_R T = \mathrm{gl.dim}\, T$.

Proof. By [3; XVI, sec. 4] we have the spectral sequence

$$H^p(S, \mathrm{Ext}^q_T(E, C)) \underset{p}{\Longrightarrow} \mathrm{Ext}^n_{S\otimes_R T}(E, C),$$

where C is an $S \otimes_R T$-module. Since S is S^e-projective, this spectral sequence collapses to $H^0(S, \mathrm{Ext}^n_T(E, C)) \approx \mathrm{Ext}^n_{S\otimes_R T}(E, C)$. From the fact that C is an arbitrary $S \otimes_R T$-module, it follows that $\mathrm{hd}_{S\otimes_R T} E \leqq \mathrm{hd}_T E$.

But considering $S \otimes_R T$ as a T-algebra, we have by [3; XVI, Exercise 5] that

$$\mathrm{hd}_T E \leqq \mathrm{hd}_T S \otimes_R T + \mathrm{hd}_{S\otimes_R T} E.$$

Since S is R-projective, it follows that $S \otimes_R T$ is T-projective. Therefore $\mathrm{hd}_T S \otimes_R T = 0$ and thus $\mathrm{hd}_T E \leqq \mathrm{hd}_{S\otimes_R T} E$, which gives the desired equality.

From the fact that $\mathrm{hd}_{S\otimes_R T} E = \mathrm{hd}_T E$ for arbitrary $S \otimes_R T$-modules E it follows that $\mathrm{gl.dim}\, S \otimes_R T \leqq \mathrm{gl.dim}\, T$. Further, if S is R-free and A is a T-module, then $\mathrm{hd}_T A = \mathrm{hd}_T S \otimes_R A$ since $S \otimes_R E$ is a direct sum of copies of A. But $\mathrm{hd}_T S \otimes_R A = \mathrm{hd}_{S\otimes_R T} S \otimes_R A$ by the previous arguments. Thus $\mathrm{hd}_T A = \mathrm{hd}_T S \otimes_R A$, which means that $\mathrm{gl.dim}\, T \leqq \mathrm{gl.dim}\, S \otimes_R T$.

COROLLARY 5.2. *Let S and T be noetherian R-algebras such that S is unramified and R-projective and $\mathfrak{J}$ is a finitely generated ideal in S^e. If T is a regular ring of finite (Krull) dimension and $S \otimes_R T$ is noetherian, then $S \otimes_R T$ is a regular ring of (Krull) dimension less than or equal to that of T.*

Further, if S is R-free (e.g. R a local ring) then the dimensions of $S \otimes_R T$ and T are equal.

Proof. By 2.5 we have that S is S^e-projective. Since T is a regular ring of finite dimension, we have by [1, Corollary 4.8] that $\mathrm{gl.dim}\, T < \infty$. Therefore it follows from 5.1 that $\mathrm{gl.dim}\, S \otimes_R T \leqq \mathrm{gl.dim}\, T$, which means that $S \otimes_R T$ is a regular ring of dimension less than or equal to that of T. The rest of the corollary follows from the fact that if $\mathrm{gl.dim}\, S \otimes_R T = \mathrm{gl.dim}\, T$, then the dimensions of $S \otimes_R T$ and T are equal.

PROPOSITION 5.3. *Let S be an R-algebra, where R is an integral domain*

with field of quotients K such that S^e is noetherian and $0 < [S \otimes_R K : K] < \infty$. Then $\mathrm{hd}_{S \otimes_R S} S = 0$ or ∞.

Proof. First we observe that $\mathrm{hd}_{S^e} S \geqq \mathrm{hd}_{L^e} L$ where $L = S \otimes K$ and $L^e = (S \otimes_R K) \otimes_K (S \otimes_R K)$. Since $[L : K] < \infty$, it is well known that if L is not a separable K-algebra, then $\mathrm{hd}_{L^e} L = \infty$. Thus we may assume that L is a separable K-algebra. Further, let us assume that S is not S^e-projective. Therefore we have that $\mathfrak{J} \neq (0)$ and the ideal generated by $\mathfrak{J}$ and $\mathfrak{N}$ is not S^e. By 2.6 we know that $\mathfrak{N} \otimes_R K$ is the annihilator of the kernel of $L^e \to L$. From the fact that L is a separable K-algebra we know that $\mathrm{hd}_{L^e} L = 0$ and hence by 2.1 $\mathfrak{N} \otimes_R K \neq (0)$. Therefore $\mathfrak{N} \neq (0)$ and thus $\mathfrak{J}$ consists entirely of zero divisors in S^e.

Let $\mathfrak{M}$ be a maximal ideal in S^e containing $\mathfrak{J}$ and $\mathfrak{N}$. Then the ideal $\mathfrak{J}_{\mathfrak{M}}$ in the local ring $S^e{}_{\mathfrak{M}}$ is not zero and consists entirely of zero-divisors. Therefore we conclude from the exact sequence

$$0 \to \mathfrak{J}_{\mathfrak{M}} \to (S^e)_{\mathfrak{M}} \to S_{\mathfrak{M}} \to 0$$

that the annihilator of $S_{\mathfrak{M}}$ as an $(S^e)_{\mathfrak{M}}$-module is not zero and consists entirely of zero-divisors. Hence by [2, 6.2] we have that the $\mathrm{hd}_{(S^e)_{\mathfrak{M}}} S_{\mathfrak{M}} = \infty$. Since $\mathrm{hd}_{S^e} S \geqq \mathrm{hd}_{(S^e)_{\mathfrak{M}}} S_{\mathfrak{M}}$, we have that $\mathrm{hd}_{S^e} S = \infty$.

Appendix.

PROPOSITION A.1. *Let R be a noetherian ring and T an R-algebra which is a finitely generated module. If S_1 and S_2 are unramified subalgebras of T, then the subalgebra generated by S_1 and S_2 is an unramified R-algebra. Thus T contains an unramified R-algebra, which contains all the unramified R-subalgebras of T.*

Proof. By 2.11 we know that $S_1 \otimes_R S_2$ is an unramified R-algebra. Therefore $Im(S_1 \otimes S_2 \to T)$ is an unramified R-subalgebra of T, which establishes the first part of the proposition. The second part follows from the chain conditions in T.

We next observe that Proposition 4.5 is true without assuming that S is a local ring. As in the proof of Proposition 4.5, it suffices to show that $t(S)$, the torsion submodule of S, is zero.

Since S is unramified, $S \otimes_R K$ is a separable K-algebra. Let $S' = Im(S \to S \otimes K)$. Then S' is a torsion-free R-algebra which is unramified since it is the image of an unramified R-algebra. Since $t(S)$ is a finitely

generated R-torsion module, there is a non-zero x in R such that $xt(S) = 0$. Now let $\mathfrak{M}$ be a maximal ideal in S. Then $t(S) \subset \mathfrak{M}$ for if not, we have that $S_{\mathfrak{M}}$ and $t(S)_{\mathfrak{M}} = x(S_{\mathfrak{M}}) = 0$ which is impossible.

Since S' is R-projective, we have that

$$0 \to t(S)/\mathfrak{m}t(S) \to S/\mathfrak{m}S \to S'/\mathfrak{m}S' \to 0$$

is exact. Thus

$$0 \to (t(S)/\mathfrak{m}t(S))_{\mathfrak{M}} \to (S'/\mathfrak{m}S)_{\mathfrak{M}} \to (S'/\mathfrak{m}S')_{\mathfrak{M}} \to 0$$

is exact. Since both S and S' are unramified, the map $(S/\mathfrak{m}S)_{\mathfrak{M}} \to (S'/\mathfrak{m}S')_{\mathfrak{M}} \to 0$ is a field epimorphism, hence an isomorphism. Thus $(t(S)/\mathfrak{m}t(S))_{\mathfrak{M}} = 0$ for all maximal ideals $\mathfrak{M}$ of S which implies that $t(S)/\mathfrak{m}t(S) = 0$. Hence $t(S) = 0$, which means that $S \approx S'$ and thus we are done.

PROPOSITION A. 2. *Let $R \subset S \subset T$ be noetherian, integrally closed domains such that T is a finitely generated R-module. Then T is unramified over R if and only if T is unramified over S and S is unramified over R.*

Proof. Suppose T is unramified over R. Then T is unramified over S. Since S is integrally closed we know by Proposition 4. 6, that T is S-projective. Thus the result follows from Proposition 4. 8.

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ON THE 14-TH PROBLEM OF HILBERT.*[1]

To Professor Oscar Zariski on his sixtieth birthday.

By Masayoshi Nagata.

The following problem is known as the 14-th problem of Hilbert:

Let k be a field and let $x_1, \cdots, x_n$ be algebraically independent elements over k. Let K be a subfield of $k(x_1, \cdots, x_n)$ containing k. Is $k[x_1, \cdots, x_n] \cap K$ finitely generated over k?

The purpose of the present paper is to answer the question in the negative by giving a counter-example. In fact, we shall give a counter-example to the following restricted case, which was the original question of Hilbert, and which we shall call the *original 14-th problem:*

Let G be a subgroup of the full linear group of $k[x_1, \cdots, x_n]$ and let o be the set of elements of $k[x_1, \cdots, x_n]$ which are invariant under G. Is o finitely generated over k?

We shall note that the construction of our example is *independent of the characteristic* (and k may be the field of complex numbers).

In § 1, we shall pass in review the history of the 14-th problem of Hilbert and shall state remaining problems concerning it. In § 2, we shall construct a counter-example and in § 3, we shall prove a lemma on plane curves which we need for the construction of our counter-example.

1. The history. The 14-th problem of Hilbert is one of the problems offered by Hilbert at the International Congress in Paris (1900) and published in *Archiv f. Math. u. Phys.* (1901) (see [1]).

It seems to the writer that no contribution to the problem was made until 1953 when Zariski proved that the answer of the 14-th problem of Hilbert is affirmative if $\dim K \leqq 2$ (under the notation stated in the introduction). Zariski proved, in fact, a more general result that the following problem, which we shall call the *generalized 14-th problem*, has affirmative answer if $\dim K \leqq 2$:

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Let $\mathfrak{o}$ *be a normal affine ring over a field* k *and let* K *be a subfield of the function field of* $\mathfrak{o}$ *containing* k. *Is* $\mathfrak{o} \cap K$ *an affine ring?* (See [2].)

In 1956, the writer gave another proof of the result of Zariski stated above, including the case where $\mathfrak{o}$ is a normal affine ring over a ground ring (see [3]).

In 1957, Rees gave a counter-example to the generalized 14-th problem in the case where $\dim K = 3$. But, the field of quotients of $\mathfrak{o} \cap K$ in his example contains the function field of a non-singular cubic curve (on a projective plane) and therefore his example cannot be a counter-example to the 14-th problem of Hilbert. (See [4].)

In 1958, the writer found at first a counter-example to the 14-th problem and then another example which is a counter-example to the original 14-th problem. This second example was announced at the International Congress in Edinburgh (1958) (see [5]). Though the first example is in the case where $\dim K = 4$, in the second example $\dim K$ is equal to 13. Then the writer noticed that the first example is also a counter-example to the original 14-th problem (and the example will be stated in the present paper).

By virtue of our example, the following two problems will be the remaining problems concerning the 14-th problem of Hilbert:

PROBLEM 1. *Let* $\mathfrak{o}$ *be a normal affine ring over a ground field* k *and let* G *be a group of automorphisms of* $\mathfrak{o}$ *over* k. *Find good sufficient conditions for the pair* $\mathfrak{o}$ *and* G *so that the set of elements of* $\mathfrak{o}$ *which are invariant under* G *forms an affine ring over* k.

PROBLEM 2. *In the 14-th problem of Hilbert, assume that* $\dim K = 3$. *What is the answer in this case?*

2. The construction of the example. Let $P_1, \cdots, P_r$ be independent generic points of the projective plane S over the prime field π of an arbitrary characteristic. We choose r so that the following is true:

(*) *If a curve* C *of degree* d *goes through every* P_i *with multiplicity at least* m (>0), *then* d/m *is greater than* $\sqrt{r}$.

The existence of such an r will be proved in § 3.

The above assumption implies the following (cf. [5]):

Let $\mathfrak{p}_i$ be the homogeneous prime ideal of P_i in a homogeneous coordinate ring $\mathfrak{H} = k[x, y, z]$ of S over a ground field k, over which the P_i are rational. Set $\mathfrak{a}_m = \cap\, \mathfrak{p}_i^m$ for every natural number m. Then

LEMMA 1. *For any natural number* m, *there exists a natural number* n *such that* $\mathfrak{a}_m{}^n \neq \mathfrak{a}_{mn}$.

Proof. Since the virtual dimension of the system of plane curves of degree d which go through every P_i with multiplicity at least m is equal to $(d^2+3d-rm^2-rm)/2$, we see that $\lim_{m\to\infty} \deg \mathfrak{a}_m/m \leqq \sqrt{r}$.[2] By the assumption (*), we have $\lim \deg \mathfrak{a}_m/m = r$. Since $\deg \mathfrak{a}_m/m > \sqrt{r}$, we see that for a sufficiently large n, $\deg \mathfrak{a}_{mn} \neq \deg \mathfrak{a}_m{}^n$ $(= n \cdot \deg \mathfrak{a}_m)$, which proves the lemma.

Now we shall construct the example. Let a_{ij} $(i=1,2,3;\ j=1,\cdots,r)$ be algebraically independent elements over the prime field and let k be a field containing all the a_{ij}. Then the points $P_i=(a_{i1},a_{i2},a_{i3})$ are independent generic points of the projective plane S over π and they are rational over k. Let $x_1,\cdots,x_r,\ y_1,\cdots,y_r$ be algebraically independent elements over k. Let V^* be the vector space of dimension r over k and let V be the subspace of V^* of dimension $r-3$ which is orthogonal to the vectors $(a_{i1},\cdots,a_{ir})$ $(i=1,2,3)$. Let G be the set of linear transformations σ of $k[x_1,\cdots,x_r, y_1,\cdots,y_r]$ such that i) $\sigma(y_i)=c_iy_i$ with $c_i\in k$ such that $c_1\cdots c_r=1$ and ii) $\sigma(x_i)=c_i(x_i+b_iy_i)$ with $(b_1,\cdots,b_r)\in V$ (and with the same c_i as in i)). Then

THEOREM. *The set* $\mathfrak{o}$ *of elements of* $k[x_1,\cdots,x_r,y_1,\cdots,y_r]$ *which are invariant under* G *is not finitely generated over* k.

Set $t=y_1\cdots y_r$, $u_i=t/y_i$, $v_i=x_iu_i$ and $w_j=\Sigma a_{ji}v_i$. We shall show at first the following

LEMMA 2. $\mathfrak{o}=k[x_1,\cdots,x_r,y_1,\cdots,y_r]\cap k(w_1,w_2,w_3,t)$.

Proof. It is sufficient to prove that the invariant subfield of $k(x_1,\cdots,x_r, y_1,\cdots,y_r)$ under G is $k(w_1,w_2,w_3,t)$. Let σ be an arbitrary element of G; let c_i and b_i be as above for this σ. $\sigma(t)=c_1\cdots c_rt=t$, hence t is invariant under G. $\sigma(w_j)=\Sigma a_{ji}(c_ix_i+c_ib_iy_i)t/c_iy_i=w_j+\Sigma a_{ji}b_it=w_j$, hence $k(w_1,w_2,w_3,t)$ is contained in the invariant subfield. Since a_{ij} are independent over π, $k(x_1,\cdots,x_r,y_1,\cdots,y_r)=k(w_1,w_2,w_3,x_4,\cdots,x_r,y_1,\cdots,y_r)$ and G is a group of linear transformation of $k[w_1,w_2,w_3,x_4,\cdots,x_r,y_1,\cdots,y_r]$: $\sigma(w_j)=w_j$, $\sigma(y_i)=c_iy_i$ with $c_i\in k$ such that $c_1\cdots c_r=1$ and $\sigma(x_i)=c_i(x_i+b_iy_i)$ (for $i\geqq 4$) with $b_i\in k$. Let G' be the set of σ in G which have c_i all equal to 1. Then we see that the invariant subfield for G' is $k(w_1,w_2,w_3,y_1,\cdots,y_r)$. G operates on $k[w_1,w_2,w_3,y_1,\cdots,y_r]$ as the group of linear transformations σ of the following type: $\sigma(w_j)=w_j$ and $\sigma(y_i)=c_iy_i$ with $c_i\in k$ such that $c_1\cdots c_r=1$. Since $t=y_1\cdots y_r$,

[2] $\deg \mathfrak{a}_m$ denotes the minimum of $\deg f$ $(f\in\mathfrak{a}_m)$.

$$k(w_1, w_2, w_3, t, y_2, \cdots, y_r) = k(w_1, w_2, w_3, y_1, \cdots, y_r)$$

and G operates on $k[w_1, w_2, w_3, t, y_2, \cdots, y_r]$ as the set G^* of linear transformations σ of the following type: $\sigma(w_j) = w_j$, $\sigma(t) = t$, $\sigma(y_i) = c_i y_i$ with arbitrary non-zero element c_i in k for $i \geqq 2$ and we see that the invariant subfield for G^*, hence for G too, is $k(w_1, w_2, w_3, t)$, which proves the lemma.

Since w_1, w_2, w_3 are algebraically independent over k, we can regard $k[w_1, w_2, w_3]$ as a homogeneous coordinate ring of the projective plane S, setting $x = w_1$, $y = w_2$, $z = w_3$. Then, using the notations at the beginning of this section,

LEMMA 3. *$\mathfrak{o}$ is the set of elements of the form $\Sigma a_n t^{-n}$ (finite sum) such that* i) *$a_n \in \mathfrak{H}$ and* ii) *if $n > 0$, then $a_n \in \mathfrak{a}_n$.*

Proof. Since a_{ij} are independent over π, we have $k[v_1, \cdots, v_r] = k[w_1, w_2, w_3, v_4, \cdots, v_r]$, which shows that

$$k[x_1, \cdots, x_r, y_1, \cdots, y_r, 1/y_1, \cdots, 1/y_r] = k[w_1, w_2, w_3, x_4, \cdots, x_r, y_1, \cdots, y_r, 1/y_1, \cdots, 1/y_r].$$

The intersection of this last ring with $k(w_1, w_2, w_3, y_1, \cdots, y_r)$ is equal to $k[w_1, w_2, w_3, y_1, \cdots, y_r, 1/y_1, \cdots, 1/y_r]$. The intersection of this last ring with $k(w_1, w_2, w_3, t)$ is equal to $k[w_1, w_2, w_3, t, 1/t]$. Hence, by virtue of Lemma 2, we see that $\mathfrak{o}$ is contained in $k[w_1, w_2, w_3, t, 1/t]$. Since $\mathfrak{H} = k[w_1, w_2, w_3]$, we have

(1) *Any element of $\mathfrak{o}$ can be expressed in the form $\Sigma a_n t^{-n}$ (finite sum) with $a_n \in \mathfrak{H}$.*

Let $\mathfrak{v}_i$ be the valuation ring $k[x_1, \cdots, x_r, y_1, \cdots, y_r]_{(y_i)}$ and let V_i be the normalized valuation defined by $\mathfrak{v}_i$. Then $\mathfrak{o}$ is contained in every $\mathfrak{v}_i$.

(2) *An element f of $\mathfrak{H}$ has value not less than m (> 0) under V_i if and only if $f \in \mathfrak{p}_i^m$ $(= \mathfrak{p}_i^{(m)})$.*

Proof. $\mathfrak{p}_i$ is generated by $z_i = a_{3i} w_1 - a_{1i} w_3$ and $z'_i = a_{3i} w_2 - a_{2i} w_3$. Obviously $V_i(z_i) = V_i(z'_i) = 1$. Furthermore, we see easily that z_i/t and z'_i/t are algebraically independent modulo the maximal ideal $\mathfrak{m}_i$ of $\mathfrak{v}_i$ over $k(w_3 \text{ modulo } \mathfrak{m}_i)$. Therefore $z_i/z'_i \text{ modulo } \mathfrak{m}_i$ is transcendental over $k(w_3 \text{ modulo } \mathfrak{m}_i)$. Hence $\mathfrak{H}[z_i/z'_i]_{\mathfrak{n}_i}$ $(\mathfrak{n}_i = \mathfrak{m}_i \cap \mathfrak{H}[z_i/z'_i])$ is a valuation ring dominated by $\mathfrak{v}_i$. Therefore (2) is proved easily.

Furthermore, by the algebraic independence of z_i/t and $z'_i/t \text{ modulo } \mathfrak{m}_i$ stated above, we have

(3) *If $a_n \in \mathfrak{H}$, then $V_i(\Sigma a_n t^{-n}) = \min V_i(a_n t^{-n})$, provided that the summation is a finite sum.*

Now we shall prove Lemma 3. If $a_n \in \mathfrak{a}_n$, then a_n is divisible by t^n in $k[x_1, \cdots, x_r, y_1, \cdots, y_r]$ and therefore any element of the form $\Sigma a_n t^{-n}$ (finite sum), with $a_n \in \mathfrak{H}$ such that if $n > 0$ then $a_n \in \mathfrak{a}_n$, is in $\mathfrak{o}$. Conversely, if c is an element of $\mathfrak{o}$, then $c = \Sigma a_n t^{-n}$ (finite sum) with $a_n \in \mathfrak{H}$ by (1) above. Since $\mathfrak{o}$ is contained in every $\mathfrak{v}_i$, $V_i(c) \geqq 0$ for any i. Therefore, by (2) and (3), we see that if $n > 0$ then $a_n \in \mathfrak{a}_n$. Thus Lemma 3 is proved.

By virtue of Lemmas 1 and 3, in order to prove Theorem, it is sufficient to prove the following

LEMMA 4. *Let $\mathfrak{b}_1, \mathfrak{b}_2, \cdots$ be a sequence of ideals in an integral domain $\mathfrak{s}$ such that* (i) $\mathfrak{b}_{i+1} \subseteq \mathfrak{b}_i$ *and* (ii) $\mathfrak{b}_i\mathfrak{b}_j \subseteq \mathfrak{b}_{i+j}$. *Let t be a transcendental element over $\mathfrak{s}$. Then the set $\mathfrak{s}'$ of elements of the form $\Sigma b_j t^{-j}$ (finite sum) with $b_j \in \mathfrak{s}$ such that if $j > 0$ then $b_j \in \mathfrak{b}_j$ forms an integral domain. If $\mathfrak{s}'$ is finitely generated over $\mathfrak{s}$, then there exists an integer m such that $\mathfrak{b}_m{}^l = \mathfrak{b}_{ml}$ for every natural number l.*[3]

Proof. It is obvious that $\mathfrak{s}'$ is an integral domain. Assume that $\mathfrak{s}'$ is finitely generated over $\mathfrak{s}$. Then there exists an s such that $\mathfrak{s}'$ is generated by t and elements of the form $b_i t^{-i}$ with $0 < i \leqq s$, $b_i \in \mathfrak{b}_i$. Let n be an arbitrary natural number. Since, for any element $b_n \in \mathfrak{b}_n$, $b_n t^{-n}$ is in $\mathfrak{s}'$, we have $\mathfrak{b}_n \subseteq \Sigma \mathfrak{b}_1{}^{\sigma_1} \cdots \mathfrak{b}_s{}^{\sigma_s}$ where the summation runs over all such that $\Sigma \sigma_j \cdot j = n$. Set $s' = s!$ and $s'' = ss'$. We consider the case where $n \geqq s''$. Since $\Sigma \sigma_j \cdot j = n \geqq s''$, there exists a j such that $\sigma_j \cdot j \geqq s'$. Then $\mathfrak{b}_1{}^{\sigma_1} \cdots \mathfrak{b}_s{}^{\sigma_s} = \mathfrak{b}_j{}^{(s'/j)} \cdot \mathfrak{b}_1{}^{\sigma_1} \cdots \mathfrak{b}_{j-1}{}^{\sigma_{j-1}} \cdot \mathfrak{b}_j{}^{\sigma_j - (s'/j)} \cdot \mathfrak{b}_{j+1}{}^{\sigma_{j+1}} \cdots \mathfrak{b}_s{}^{\sigma_s} \subseteq \mathfrak{b}_{s'}\mathfrak{b}_{n-s'}$. It follows that $\mathfrak{b}_n \subseteq \mathfrak{b}_{s'}\mathfrak{b}_{n-s'}$ and therefore

If $n \geqq s''$, then $\mathfrak{b}_n = \mathfrak{b}_{s'}\mathfrak{b}_{n-s'}$.

Now we consider the case where n is an arbitrary multiple of s'': $n = s''l$. Then the above result shows that $\mathfrak{b}_{s''l} = \mathfrak{b}_{s'}{}^{s(l-1)}\mathfrak{b}_{s''}$. Hence $\mathfrak{b}_{s''l} \subseteq \mathfrak{b}_{s''}{}^l$, whence $\mathfrak{b}_{s''l} = \mathfrak{b}_{s''}{}^l$. This proves Lemma 4 with $m = s''$.

Remark. We can prove Theorem without proving Lemma 4, by the same method as in [5].

3. The existence of r.

PROPOSITION. *If r is the square of a natural number s not less than 4, then r satisfies the requirement in* § 1.

Proof. (i) The case where r is odd. Set $s' = (s+1)/2$ and let C_s, $C_{s'}$, $C'_{s'}$ be independent generic curves of degree s, s', s' respectively. Let $P_1, \cdots, P_s$

[3] This lemma was substantially proved by Rees [4].

be s points among $C_{s'} \cdot C'_{s'}$. Then the P_i are independent generic points of the projective plane over $\pi(C_s)$. Let $P^*_1, \cdots, P^*_s$ be independent generic points of C_s over $\pi(C_s)$, let $C^*_{s'}$ and $C^{*\prime}_{s'}$ be most general curves of degree s' going through the P^*_i, and let $Q^*_1, \cdots, Q^*_{r-s}$ be such that

$$C^*_{s'} \cdot C_s = \Sigma P^*_i + \Sigma_1^{(r-s)/2} Q^*_i, \quad C^{*\prime}_{s'} \cdot C_s = \Sigma P^*_i + \Sigma_{(r-s+2)/2}^{r-s} Q^*_i.$$

We consider a specialization

$$(P_1, \cdots, P_s, C_{s'}, C'_{s'}) \to (P^*_1, \cdots, P^*_s, C^*_{s'}, C^{*\prime}_{s'})$$

over $\pi(C_s)$. We take $Q_1, \cdots, Q_{r-s}$ so that (1) $\Sigma_1^{(r-s)/2} Q_i \subseteq C_{s'} \cdot C_s$, $\Sigma_{(r-s+2)/2}^{r-s} Q_i \subseteq C'_{s'} \cdot C_s$ and (2) the Q_i are specialized to the Q^*_i by the specialization considered above.

Assume now that for an m there exists a curve of degree sm which goes through each of given r $(= s^2)$ independent generic points of the projective plane S with multiplicity at least m. Then we see that there exists a curve E of degree sm which goes through the Q_i and the P_i with multiplicity at least m. Assume that E does not contain $C_{s'}$ as a component. Then $C_{s'} \cdot E$ contains $\Sigma mP_i + \Sigma_1^{(r-s)/2} mQ_i$. Since $\deg C_{s'} \cdot E = ss'm = s(s+1)m/2$, we have $C_{s'} \cdot E = \Sigma mP_i + \Sigma_1^{(r-s)/2} mQ_i$. Since $s \geqq 5$, $s' \geqq 3$, hence $C_{s'}$ is of positive genus. Since the P_i are independent generic points of $C_{s'}$ over $\pi(C_{s'}, Q_1, \cdots, Q_{(r-s)/2})$, we have a contradiction by the following obvious

LEMMA 5. *If C is a plane curve of positive genus and if $R_1, \cdots, R_n$ are points of C such that some of the R_i are independent generic points of C over a field of definition of C and the other R_i's, then for any natural numbers $c_1, \cdots, c_n$, there exists no curve whose intersection with C is equal to $\Sigma c_i R_i$.*

Therefore E must contain $C_{s'}$ as a component. Similarly, E must contain $C'_{s'}$ as a component. Then, specializing $E - C_{s'} - C'_{s'}$ over the specialization we considered above, we see the existence of a curve E^* of degree $s(m-1)-1$ which goes through the P^*_i with multiplicity at least $m-2$ and the Q^*_j with multiplicity at least $m-1$. Hence, if $m = 1$, we have a contradiction. We shall use the induction on m. Assume that E^* does not contain C_s as a component. Then $C_s \cdot E^*$ contains $\Sigma(m-2)P^*_i + \Sigma(m-1)Q^*_j$. By the equality of the degrees, we have $C_s \cdot E^* = \Sigma(m-2)P^*_i + \Sigma(m-1)Q^*_j$. Since $C_s(C_{s'} + C'_{s'}) = \Sigma 2P^*_i + \Sigma Q^*_j$, we have $C_s \cdot ((m-1)(C_{s'} + C'_{s'}) - E^*) = \Sigma mP^*_i$, which gives a contradiction by Lemma 5. Thus E^* must contain C_s as a component and we have a contradiction by induction on m.

(ii) The case where r is even. Set $s' = (s+2)/2$ and let C_s, $C_{s'}$, $C'_{s'}$ be independent generic curves of degree s, s', s' respectively. Let $P_1, \cdots, P_{2s}$

be $2s$ points among $C_{s'} \cdot C'_{s'}$. Then we take independent generic points $P^*_1, \cdots, P^*_{2s}$ of C_s over $\pi(C_s)$. Then we prove the assertion by the same way as in (i) and we omit the detail (cf. [5]).

Remark. In order to satisfy the requirement in § 1 for r, r must be greater than 9.

Proof. If $r=1$ or 2, then there is a line going through P_i; if $r=3$, the sum of 3 lines going through two of the P_i gives a counter-example to the requirement; if $r=4$ or 5, then there is a conic going through the P_i; if $r=6$, the sum of 6 conics going through 5 of the P_i gives a counter-example; if $r=7$, sum of 7 cubics each of which has a double point at one of the P_i and goes through all the P_i gives a counter-example; if $r=8$, the sum of 8 curves of degree 6 each of which has a triple point at one of the P_i and has double points at all the other P_i gives a counter-example; if $r=9$, then there is a cubic going through all the P_i.

The writer has the following conjecture:

CONJECTURE. *For the requirement in* § 1 *for r, it will be enough for r to be greater than* 9.

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A NOTE ON OBSTRUCTIONS AND CHARACTERISTIC CLASSES.*

By MICHEL A. KERVAIRE.

The present paper is a generalization of [7]. Relations will be established between the obstructions associated with cross-sections in a stable $\boldsymbol{U}(n)$, $\boldsymbol{SO}(n)$ or $\boldsymbol{Sp}(n)$-bundle over a *complex* K and the characteristic classes of such bundles.

In the $\boldsymbol{U}(n)$-case, we obtain as a corollary a theorem of F. Peterson [10] stating that a $\boldsymbol{U}(n)$-bundle over a torsion free complex K of dimension $\leqq 2n$ is trivial if and only if the Chern classes of the bundle vanish.

A similar statement in the $\boldsymbol{SO}(n)$ or $\boldsymbol{Sp}(n)$ case, involving the Pontryagin, resp. symplectic Pontryagin classes, would be wrong. In case of an $\boldsymbol{SO}(n)$ [resp. $\boldsymbol{Sp}(n)$] bundle there are obstructions in $H^{8s+1}(K;\boldsymbol{Z}_2)$ and $H^{8s+2}(K;\boldsymbol{Z}_2)$ [resp. $H^{8s+5}(K;\boldsymbol{Z}_2)$ and $H^{8s+6}(K;\boldsymbol{Z}_2)$] which are not expressible in terms of characteristic classes of the bundle (see Lemma 4.3 for a precise statement). The information about these obstructions is still very poor.

In [10], F. Peterson deduces his theorem from a computation of the Postnikov decomposition of $B_{\boldsymbol{U}(n)}$. We proceed the other way around and obtain the Postnikov decomposition of $B_{\boldsymbol{U}(n)}$, $B_{\boldsymbol{SO}(n)}$ and $B_{\boldsymbol{Sp}(n)}$ in the stable range from the main lemma (Lemma 1.1).

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1. Let $\boldsymbol{G}$ be one of the groups $\boldsymbol{U}(n)$, $\boldsymbol{SO}(n)$ or $\boldsymbol{Sp}(n)$. Let ξ be a stable principal $\boldsymbol{G}$-bundle over a CW-complex K (stability means that the homotopy groups $\pi_{q-1}(\boldsymbol{G})$ are stable for $q \leqq \dim K$). Assume that ξ admits a cross-section f over the $(q-1)$-skeleton $K^{(q-1)}$ of K. Take q to be even $=2r$ if $\boldsymbol{G}=\boldsymbol{U}(n)$ and q divisible by 4, $q=4k$, if $\boldsymbol{G}=\boldsymbol{SO}(n)$ or $\boldsymbol{Sp}(n)$. Then, by [4], $\pi_{q-1}(\boldsymbol{G}) \approx \boldsymbol{Z}$ in all cases. The obstruction class $o(\xi,f) \in H^q(K;\pi_{q-1}(\boldsymbol{G}))$ to extending f over the q-skeleton can be regarded up to sign as an integer class. Denote by $c_r(\xi)$, $p_k(\xi)$, $e_k(\xi)$ the Chern class, the Pontryagin or the symplectic Pontryagin class[1] of ξ in dimension q according as $\boldsymbol{G}=\boldsymbol{U}(n)$, $\boldsymbol{SO}(n)$ or $\boldsymbol{Sp}(n)$ respectively.

* Received October 5, 1958.

[1] See [3], 9.6, for the definition.

LEMMA 1.1. *The characteristic classes $c_r(\xi)$, $p_k(\xi)$, $e_k(\xi)$ are given by the formulae*

(i) $c_r(\xi) = (r-1)!\,o(\xi,f)$ *if* $\boldsymbol{G}=\boldsymbol{U}(n)$,

(ii) $p_k(\xi) = (2k-1)!\,a_k o(\xi,f)$ *if* $\boldsymbol{G}=\boldsymbol{SO}(n)$,

(iii) $e_k(\xi) = (2k-1)!\,b_k o(\xi,f)$ *if* $\boldsymbol{G}=\boldsymbol{Sp}(n)$,

where, as in [6], $a_k \cdot b_k = 2$ *and* a_k *is equal to* 1 *for* k *even and to* 2 *for* k *odd.*

Proof. Let $\boldsymbol{G}=\boldsymbol{U}(n)$. Denote by ξ' the associated bundle with fibre $W_{n,n-r+1}=\boldsymbol{U}(n)/\boldsymbol{U}(r-1)$. Let $q\colon \boldsymbol{U}(n)\to\boldsymbol{U}(n)/\boldsymbol{U}(r-1)$ be the natural projection and q' the induced map of the total space of ξ into the total space of ξ'. The map $f'=q'\circ f$ is a cross-section of ξ' restricted to the $(q-1)$-skeleton. Denote by $q_*\colon \pi_{2r-1}(\boldsymbol{U}(n))\to\pi_{2r-1}(W_{n,n-r+1})$ and

$$q_{**}\colon H^{2r}(K;\pi_{2r-1}(\boldsymbol{U}(n)))\to H^{2r}(K;\pi_{2r-1}(W_{n,n-r+1}))$$

the homomorphisms induced by q. Clearly, $q_{**}o(\xi,f)=o(\xi',f')=$ the obstruction to extending f' over the $2r$-skeleton. In other words $q_{**}o(\xi,f)=c_r(\xi)$. Identifying $\pi_{2r-1}(\boldsymbol{U}(n))$ and $\pi_{2r-1}(W_{n,n-r+1})$ with $\boldsymbol{Z}$ (disregarding signs), we have $q_{**}u=(r-1)!\,u$ for any $u\in H^*(K;\boldsymbol{Z})$ because q_* maps a generator of $\pi_{2r-1}(\boldsymbol{U}(n))$ onto $(r-1)!$ times a generator of $\pi_{2r-1}(W_{n,n-r+1})$ according to [5]. Thus $c_r(\xi)=q_{**}o(\xi,f)=\pm(r-1)!\,o(\xi,f)$.

The proofs of (ii) and (iii) are entirely similar and are left to the reader (compare also [9]).

2. As a corollary we obtain the

THEOREM 2.1 (F. Peterson). *Let ξ be a $\boldsymbol{U}(n)$-bundle over a complex K with* $\dim K\leqq 2n$ *and assume that $H^{2r}(K;\boldsymbol{Z})$ has no torsion except possibly prime to $(r-1)!$ for $r=1,2,\cdots$. Then ξ is trivial if and only if the Chern classes $c_1,\cdots,c_n$ vanish.*

Proof. Half of the statement is trivial. We prove that ξ is the product bundle provided $c_1(\xi)=0$, $c_2(\xi)=0,\cdots,c_n(\xi)=0$ by stepwise extension of a cross-section in the associated principal bundle ξ_P.

If f is a cross-section in ξ_P restricted to $K^{(q-1)}$ and q is odd, there is no obstruction to extending f to $K^{(q)}$ since $\pi_{2i}(\boldsymbol{U}(n))=0$ for $i<n$ by [4]. Let q be even: $q=2r$. Then by Lemma 1.1 the obstruction class $o(\xi,f)$ satisfies the identity $c_r=\pm(r-1)!\,o(\xi,f)$. Under the assumptions of the theorem this implies $o(\xi,f)=0$. It follows (see [11], 34.2) that $f\mid K^{(q-2)}$ is extendable over $K^{(q)}$. This proves the theorem by induction on q.

Some information on the obstructions arising in the $\boldsymbol{SO}(n)$ and $\boldsymbol{Sp}(n)$ cases is given in Lemmas 4.1 and 4.2 below. We need a preliminary lemma.

3. Let $\boldsymbol{G}$ be any Lie group and $\boldsymbol{H}$ a closed subgroup of $\boldsymbol{G}$ such that the sequence

$$0 \to \pi_q(\boldsymbol{G}/\boldsymbol{H}) \xrightarrow{\partial} \pi_{q-1}(\boldsymbol{H}) \xrightarrow{i_*} \pi_{q-1}(\boldsymbol{G}) \to 0 \tag{3.1}$$

is exact for some q (here i_* is induced by the inclusion $i: \boldsymbol{H} \to \boldsymbol{G}$). Assume that the $\boldsymbol{G}$-bundle ξ over the complex K admits a cross-section f over the $(q-1)$-skeleton. Let $\mathfrak{o}(\xi, f) \in H^q(K; \pi_{q-1}(\boldsymbol{G}))$ be the obstruction class to extending f over $K^{(q)}$. We want to compute $\delta^*\mathfrak{o}(\xi, f)$, where δ^* is the boundary homomorphism $\delta^*: H^q(K; \pi_{q-1}(\boldsymbol{G})) \to H^{q+1}(K; \pi_q(\boldsymbol{G}/\boldsymbol{H}))$ of the cohomology exact sequence associated with the coefficient sequence (3.1). (Compare Steenrod [11], 38.5.)

Let ξ' be the associated bundle with fibre $\boldsymbol{G}/\boldsymbol{H}$. The cross-section f induces a cross-section f' of ξ' restricted to the $(q-1)$-skeleton.

LEMMA 3.2. *Under the above exactness assumption of* (3.1), *the cross-section f' is always extendable to a cross-section F' of ξ' restricted to $K^{(q)}$. Let $\mathfrak{o}(\xi', F') \in H^{q+1}(K; \pi_q(\boldsymbol{G}/\boldsymbol{H}))$ be the obstruction class to extending F' over $K^{(q+1)}$. Then $\delta^*\mathfrak{o}(\xi, f) = \mathfrak{o}(\xi', F')$.*

Proof. Let $p: \pi_{q-1}(\boldsymbol{G}) \to \pi_{q-1}(\boldsymbol{G}/\boldsymbol{H})$ be induced by the projection $\boldsymbol{G} \to \boldsymbol{G}/\boldsymbol{H}$, and $p_*: Z^q(K; \pi_{q-1}(\boldsymbol{G})) \to Z^q(K; \pi_{q-1}(\boldsymbol{G}/\boldsymbol{H}))$ be the homomorphism induced by the coefficient homomorphism p. Let $z \in \mathfrak{o}(\xi, f)$ be the obstruction *cocycle* to extending f over $K^{(q)}$ and z' be the obstruction cocycle to extending f' over $K^{(q)}$. We have $p_*z = z'$ and since p is zero, it follows $z' = 0$. In other words, f' can be extended to a cross-section F' of ξ' restricted to $K^{(q)}$. The map $F': K^{(q)} \to E'$, where E' is the total space of ξ' induces over $K^{(q)}$ an $\boldsymbol{H}$-bundle η (E' is the quotient of the total space E of ξ by the action of $\boldsymbol{H}$ as a subgroup of $\boldsymbol{G}$). $f_\eta(x) = (x, f(x))$ for $x \in K^{(q-1)}$ defines a cross-section of η restricted to $K^{(q-1)}$ (compare Steenrod [11], 10.2). Let $u_\eta \in Z^q(K^{(q)}; \pi_{q-1}(\boldsymbol{H}))$ be the obstruction cocycle to extending f_η over $K^{(q)}$ and let $u \in C^q(K; \pi_{q-1}(\boldsymbol{H}))$ be defined by $u[\tau] = u_\eta[\tau]$ for every q-cell $\tau \in K^{(q)} \subset K$. Clearly, $i_{**}u = z$, where $i_{**}: C^q(K; \pi_{q-1}(\boldsymbol{H})) \to C^q(K; \pi_{q-1}(\boldsymbol{G}))$ is induced by $i_*: \pi_{q-1}(\boldsymbol{H}) \to \pi_{q-1}(\boldsymbol{G})$. The assertion of the lemma can be stated as

$$\delta u = \partial_* z', \tag{3.3}$$

where $\partial_*: Z^{q+1}(K; \pi_q(\boldsymbol{G}/\boldsymbol{H})) \to Z^{q+1}(K; \pi_{q-1}(\boldsymbol{H}))$ is induced by $\partial: \pi_q(\boldsymbol{G}/\boldsymbol{H})$

$\to \pi_{q-1}(\boldsymbol{H})$, and z' is the obstruction to extending F' over $K^{(q+1)}$. (Compare Steenrod [11], 38.5 formula (8).)

Let $\Phi: B^{q+1} \to K$ be the attaching map of a $(q+1)$-cell σ of K, and denote by ϕ the restriction of Φ to the boundary S^q of B^{q+1}. Then $F' \circ \phi: S^q \to E'$ induces over S^q an $\boldsymbol{H}$-bundle whose characteristic map $\chi = \Delta\alpha$ is the image of the element $\alpha \in \pi_q(E')$ represented by $F' \circ \phi$ under the boundary operator $\Delta: \pi_q(E') \to \pi_{q-1}(H)$ of the homotopy sequence of $H \to E \xrightarrow{\pi} E'$. Since $F' \circ \phi(S^q) \subset \rho^{-1}(\Phi B)$, where ρ is the projection $\rho: E' \to K$ of the bundle ξ', $F' \circ \phi$ as a map in $\rho^{-1}(\Phi B)$ represents an element of $\pi_q(G/H)$ which is equal to $z'[\sigma]$. Thus $j'_*(z'[\sigma]) = \alpha$, where $j': \boldsymbol{G/H} \to E'$ is the inclusion of the fibre. On the other hand χ is equal to $u[\phi S^q]$. Consider the commutative diagram

$$\begin{array}{ccc} \pi_q(\boldsymbol{G/H}) & \xrightarrow{\partial} & \pi_{q-1}(\boldsymbol{H}) \\ \downarrow{\scriptstyle j'_*} & & \| \\ \pi_q(E') & \xrightarrow{\Delta} & \pi_{q-1}(\boldsymbol{H}) \end{array}$$

induced by the bundle map

$$\begin{array}{ccc} \boldsymbol{G} & \xrightarrow{j} & E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ \boldsymbol{G/H} & \xrightarrow{j'} & E'. \end{array}$$

We have

$$\delta u[\sigma] = u[\phi S^q] = \Delta\alpha = \Delta j'_*(z'[\sigma]) = \partial_*(z'[\sigma]) = (\partial_* z')[\sigma].$$

Since this is true for any $(q+1)$-cell σ, the proof of (3.3) is complete.

4. We apply this to the cases $\boldsymbol{G} = \boldsymbol{SO}(2n)$ and $\boldsymbol{G} = \boldsymbol{Sp}(n)$, $\boldsymbol{H} = \boldsymbol{U}(n)$.

LEMMA 4.1. *Let the stable principal $\boldsymbol{SO}(2n)$-bundle ξ admit a cross-section f over the $(8s+1)$-skeleton $K^{(8s+1)}$ of the base complex K. Let $\mathfrak{o}(\xi, f) \in H^{8s+2}(K; \mathbf{Z}_2)$ be the obstruction class* (compare Bott [4]). *The induced cross-section f' of the associated bundle ξ' with fibre $\boldsymbol{SO}(2n)/\boldsymbol{U}(n)$ restricted to $K^{(8s+1)}$ is extendable over $K^{(8s+2)}$ to a cross-section F'. One has $\beta\mathfrak{o}(\xi, f) = \pm\, \mathfrak{o}(\xi', F')$, where β is the Bockstein operation and $\mathfrak{o}(\xi', F')$ is the obstruction class to extending F' over $K^{(8s+3)}$.*

Proof. Take $\boldsymbol{G} = \boldsymbol{SO}(2n)$ and $\boldsymbol{H} = \boldsymbol{U}(n)$ in Lemma 3.2. The sequence

$$0 \to \pi_{8s+2}(\boldsymbol{SO}(2n)/\boldsymbol{U}(n)) \to \pi_{8s+1}(\boldsymbol{U}(n)) \to \pi_{8s+1}(\boldsymbol{SO}(2n)) \to 0$$

is exact and reads $0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}_2 \to 0$. (Compare Bott [4].) The associated coboundary homomorphism δ^* is β by definition. Similarly,

LEMMA 4.2. *Let the stable principal $\boldsymbol{Sp}(n)$-bundle ξ admit a cross-section f over the $(8s+5)$-skeleton of the base complex K. Denote by $o(\xi, f) \in H^{8s+6}(K; \mathbf{Z}_2)$ the obstruction class. f induces a cross-section f' of the associated bundle ξ' with fibre $\boldsymbol{Sp}(n)/\boldsymbol{U}(n)$ restricted to $K^{(8s+5)}$. Then f' is extendable over the $(8s+6)$-skeleton. Let F' be an extension and $o(\xi', F') \in H^{8s+7}(K; \mathbf{Z})$ the obstruction class (up to sign). We have $\beta o(\xi, f) = \pm\, o(\xi', F')$.*

Proof. The sequence

$$0 \to \pi_{8s+6}(\boldsymbol{Sp}(n)/\boldsymbol{U}(n)) \to \pi_{8s+5}(\boldsymbol{U}(n)) \to \pi_{8s+5}(\boldsymbol{Sp}(n)) \to 0$$

is exact and reads $0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}_2 \to 0$. (Compare Bott [4].)

Next we show that the Stiefel-Whitney class w_q of an $\boldsymbol{SO}(n)$-bundle ξ is in general independent of the obstruction to extending over $K^{(q)}$ a cross-section of ξ restricted to $K^{(q-1)}$. In fact:

LEMMA 4.3. *If there exists a cross-section of the $\boldsymbol{SO}(n)$-bundle ξ restricted to the $(q-1)$-skeleton of the base complex K, $q > 0$ and $q \neq 2$, 4, 8, then the Stiefel-Whitney class w_q of ξ is zero.*

Proof. Let $o(\xi, f)$ be the cohomology class of the obstruction to extending the given cross-section f over the q-skeleton. Let $p: \boldsymbol{SO}(n) \to V_{n,n-q+1} = \boldsymbol{SO}(n)/\boldsymbol{SO}(q-1)$ be the natural projection, and

$$p_*: \pi_{q-1}(\boldsymbol{SO}(n)) \to \pi_{q-1}(V_{n,n-q+1}),$$

$$p_{**}: H^q(K; \pi_{q-1}(\boldsymbol{SO}(n))) \to H^q(K; \pi_{q-1}(V_{n,n-q+1}))$$

the induced homomorphisms. Clearly, $p_{**} o(\xi, f) = w_q$. The lemma follows from the fact that p_* is zero provided $q \neq 2, 4, 8$. This is trivial for $q \equiv 3$, 5, 6, 7 modulo 8 ($\pi_{q-1}(\boldsymbol{SO}(n))$ is zero in these cases by [4]). It is also trivial for $q \equiv 1 \bmod 8$ since $\pi_0(\boldsymbol{SO}(n)) = 0$ and $\pi_{8s}(\boldsymbol{SO}(n)) = \mathbf{Z}_2$, $\pi_{8s}(V_{n,n-8s}) \approx \mathbf{Z}$ for $s > 0$. For $q \equiv 2 \bmod 8$, $p_* = 0$ follows easily from considering the homotopy exact sequence of $\boldsymbol{SO}(n)/\boldsymbol{SO}(q-1)$. For $q \equiv 4 \bmod 8$, $p_* = 0$ was stated and proved in [6], Lemma 3. The case $q \equiv 0 \bmod 8$ will be treated in a forth coming paper [8].[2]

Using the Lemma 1.1 above, we obtain the k-invariants of the classifying spaces $B_{\boldsymbol{U}(n)}$, $B_{\boldsymbol{SO}(n)}$, $B_{\boldsymbol{Sp}(n)}$ in the *stable range*. (For the k-invariants of

[2] (Added in proof) $p_* = 0$ also follows from comparison with $p'_*: \pi_{q-1}(\boldsymbol{SO}(q)) \to \pi_{q-1}(S^{q-1})$ as in the proof of Lemma 6.4 below.

$B_{U(n)}$, compare F. Peterson [10]). We need a few probably well known lemmas about k-invariants which we derive in the next section.

5. Let X be a simply connected space and $X_q \supset X$ such that

(i) (X_q, X) is a relative CW-complex (compare G. W. Whitehead [12]),

(ii) $\pi_i(X_q) = 0$ for $q < i$,

(iii) $\pi_i(X) \approx \pi_i(X_q)$ for $i \leqq q$ under inclusion.

Then, $\pi_i(X_q, X) = 0$ for $i \leqq q+1$ and since $\pi_1(X) = 0$ by assumption, it follows from the relative Hurewicz theorem that $H_i(X_q, X) = 0$ for $i \leqq q+1$, and $\pi_{q+1}(X) \approx \pi_{q+2}(X_q, X) \approx H_{q+2}(X_q, X)$. It follows that $H^{q+2}(X_q, X; \pi_{q+1}) = \mathrm{Hom}(H_{q+2}(X_q, X), \pi_{q+1})$ contains a fundamental class u (π_{q+1} denotes $\pi_{q+1}(X)$ for brevity).

LEMMA 5.1. *$a^*u = k^{q+2} \in H^{q+2}(X_q; \pi_{q+1})$ is the $(q+2)$-dimensional k-invariant of X, where $a^*: H^*(X_q, X; \pi_{q+1}) \to H^*(X_q; \pi_{q+1})$ is induced by the inclusion $a: (X_q, 0) \to (X_q, X)$.*

In fact the lemma is a special case of the following definition of the characteristic class.

Let $p: E \to B$ be a fibering with q-connected fibre F, and let $i: F \to E$ be the inclusion of the fibre. Assume that E (and thus also B) is simply connected. By the homotopy exact sequence, $p_*: \pi_i(E) \to \pi_i(B)$ is an isomorphism for $i \leqq q$, and $p_*: \pi_{q+1}(E) \to \pi_{q+1}(B)$ is surjective. Therefore, $\pi_i(B', E) = 0$ for $i \leqq q+1$, where B' is the mapping cylinder of p (B and B' have the same homotopy type). Since $\pi_1(E) = 0$, it follows $H_i(B', E) = 0$ for $i \leqq q+1$, and $\pi_{q+2}(B', E) \approx H_{q+2}(B', E)$. Thus $H^{q+2}(B', E; \pi_{q+2}(B', E)) = \mathrm{Hom}(H_{q+2}(B', E), \pi_{q+2}(B', E))$ contains a fundamental class u.

Define a homomorphism $\phi: \pi_i(F) \to \pi_{i+1}(B', E)$ for every i as follows: If $f: S^i \to F$ represents a class $\alpha \in \pi_i(F)$, the formula

$$f'(x, t) = (f(x), t),$$

$0 \leqq t \leqq 1$ defines a mapping of the cone over S^i into B'. The boundary of this cone is mapped into $F \subset E$. Let $\phi\alpha$ be the class of f' in $\pi_{i+1}(B', E)$.

LEMMA 5.1′. *ϕ is an isomorphism, and $\phi_* c = a^*u$, where ϕ_* is induced by the coefficient homomorphism $\phi: \pi_{q+1}(F) \to \pi_{q+2}(B', E)$; c is the characteristic class and a^* is induced by the inclusion $(B', 0) \subset (B', E)$.*

Proof. Consider the diagram

$$\begin{array}{ccccccccc}
\pi_{i+1}(E) & \longrightarrow & \pi_{i+1}(E,F) & \xrightarrow{\partial} & \pi_i(F) & \xrightarrow{i_*} & \pi_i(E) & \longrightarrow & \pi_i(E,F) \\
\downarrow{\scriptstyle \mathrm{id.}} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathrm{id.}} & & \downarrow{\scriptstyle p_*} \\
\pi_{i+1}(E) & \xrightarrow{p_*} & \pi_{i+1}(B) & \xrightarrow{a_*} & \pi_{i+1}(B',E) & \xrightarrow{\partial'} & \pi_i(E) & \xrightarrow{p_*} & \pi_i(B).
\end{array}$$

It is easily seen from the definition of ϕ that commutativity holds in each square. Since p_* is an isomorphism, ϕ is an isomorphism by the 5-Lemma.

Let $\psi: H_i(F) \to H_{i+1}(B',E)$ be defined similarly to ϕ. The cycle z being a representative of a class $\zeta \in H_i(F)$, $\psi\zeta$ is the class in $H_{i+1}(B',E)$ of the cone over z in B', regarded as a cycle modulo E. Clearly, the following diagram is commutative:

$$\begin{array}{ccccccccc}
H_{i+1}(E) & \longrightarrow & H_{i+1}(E,F) & \xrightarrow{\partial} & H_i(F) & \xrightarrow{i_*} & H_i(E) & \longrightarrow & H_i(E,F) \\
\downarrow{\scriptstyle \mathrm{id.}} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \mathrm{id.}} & & \downarrow{\scriptstyle p_*} \\
H_{i+1}(E) & \longrightarrow & H_{i+1}(B) & \xrightarrow{a_*} & H_{i+1}(B',E) & \xrightarrow{\partial'} & H_i(E) & \longrightarrow & H_i(B).
\end{array}$$

It follows by a standard argument that the dual diagram with coefficient group $\pi_j(F) = \pi_{j+1}(B',E)$, identified by $\phi: \pi_j(F) \to \pi_{j+1}(B',E)$, is also commutative. In particular,

$$\begin{array}{ccc}
H^{q+2}(E,F;\pi_{q+1}) & \xleftarrow{\delta} & H^{q+1}(F;\pi_{q+1}) \\
\uparrow{\scriptstyle p^*} & & \uparrow{\scriptstyle \psi^*} \\
H^{q+2}(B;\pi_{q+1}) & \xleftarrow{a^*} & H^{q+2}(B',E;\pi_{q+1})
\end{array}$$

is commutative (π_{q+1} denotes $\pi_{q+1}(F)$ identified with $\pi_{q+2}(B',E)$ by ϕ). Since c is characterized by $p^*c = \delta v$, where v is the fundamental class in $H^{q+1}(F;\pi_{q+1})$, it remains only to prove that $\psi^* u = v$. This is obvious, and the proof of Lemma 5.1′ is complete.

Consider the cohomology exact sequence

$$\cdots \to H^{q+1}(X;\pi_{q+1}) \xrightarrow{\delta} H^{q+2}(X_q, X;\pi_{q+1}) \xrightarrow{a^*} H^{q+2}(X_q;\pi_{q+1}) \to \cdots$$

Clearly, δ annihilates every decomposable element of $H^{q+1}(X;\pi_{q+1})$.

In fact, if $c \in H^{q+1}(X;\pi_{q+1})$ is any class, δc as a homomorphism of π_{q+1} into π_{q+1} is given by

$$\delta c[\alpha] = \alpha^* c, \text{ for every } \alpha \in \pi_{q+1}, \tag{5.2}$$

where $\alpha^*: H^{q+1}(X;\pi_{q+1}) \to H^{q+1}(S^{q+1};\pi_{q+1})$ is induced by any map $S^{q+1} \to X$ representing α, and $H^{q+1}(S^{q+1};\pi_{q+1})$ is identified with π_{q+1}.

6. As an application of Lemma 1.1 and the preceding remarks, we obtain:

THEOREM 6.1. *Let* $X = B_{U(n)}$ *and* $\pi_{q+1} = \pi_{q+1}(B_{U(n)}) \approx \pi_q(\boldsymbol{U}(n))$. *Then, for* $i < n$, *we have* $\boldsymbol{k}^{2i+2} = 0$, *and* X_{2i+1} *may be taken equal to* X_{2i}. $H^{2i+3}(X_{2i+1}; \pi_{2i+2}) = H^{2i+3}(X_{2i+1}; \mathbf{Z})$ *is a finite cyclic group of order* $i!$ *generated by the k-invariant* $\boldsymbol{k}^{2i+3}$.

THEOREM 6.2. *Let* $X = B_{SO(n)}$ *and* $\pi_{q+1} = \pi_{q+1}(B_{SO(n)}) \approx \pi_q(\boldsymbol{SO}(n))$. *Then, in the stable range,* $\boldsymbol{k}^{4j} = 0$, $\boldsymbol{k}^{8i-2} = 0$, $\boldsymbol{k}^{8i-1} = 0$. *One can take* $X_{4j-1} = X_{4j-2}$, $X_{8i-2} = X_{8i-3} = X_{8i-4}$. *The k-invariants* $\boldsymbol{k}^{4j+1}$, $\boldsymbol{k}^{8i+2}$, $\boldsymbol{k}^{8i+3}$ *are different from zero. Specifically, we have exact sequences*

$$(6.2') \qquad 0 \to \mathbf{Z}_{(2j-1)!a_j} \xrightarrow{\tilde{a}^*} H^{4j+1}(X_{4j-1}; \pi_{4j}) \to H^{4j+1}(X; \pi_{4j}) \to 0$$

which split for $j \geqq 3$. *The k-invariant* $\boldsymbol{k}^{4j+1}$ *is the image under* $\tilde{a}^*$ *of a generator of* $\mathbf{Z}_{(2j-1)!a_j}$. *For* $j = 1$ *or* 2, $\boldsymbol{k}^{4j+1}$ *can be halved and* $\frac{1}{2}\boldsymbol{k}^{4j+1}$ *can be chosen so as to project onto* $W^{4j+1} \in H^{4j+1}(X; \pi_{4j})$. *Similarly, the sequences*

$$(6.2'') \qquad \begin{aligned} 0 \to \mathbf{Z}_2 &\xrightarrow{a^*} H^{8i+2}(X_{8i}; \pi_{8i+1}) \to H^{8i+2}(X; \pi_{8i+1}) \to 0, \\ 0 \to \mathbf{Z}_2 &\xrightarrow{a^*} H^{8i+3}(X_{8i+1}; \pi_{8i+2}) \to H^{8i+3}(X; \pi_{8i+2}) \to 0, \end{aligned}$$

are exact and split. $\boldsymbol{k}^{8i+2}$, *resp.* $\boldsymbol{k}^{8i+3}$ *are the images under* a^* *of the generator of* $\mathbf{Z}_2$. ($\pi_{4j} \approx \mathbf{Z}$, $\pi_{8i+1} \approx \pi_{8i+2} \approx \mathbf{Z}_2$, by [4].)

THEOREM 6.3. *Let* $X = B_{Sp(n)}$ *and* $\pi_{q+1} = \pi_{q+1}(B_{Sp(n)}) \approx \pi_q(\boldsymbol{Sp}(n))$. *Then, in the stable range,* $\boldsymbol{k}^{4j} = 0$, $\boldsymbol{k}^{8i+2} = 0$, $\boldsymbol{k}^{8i+3} = 0$. *One can take* $X_{4j-1} = X_{4j-2}$, $X_{8i+2} = X_{8i+1} = X_{8i}$. *The other k-invariants are non-zero. Precisely,*

$$H^{4j+1}(X_{4j-1}; \mathbf{Z}) \approx \mathbf{Z}_{(2j-1)!b_j} \text{ generated by } \boldsymbol{k}^{4j+1},$$

$$H^{8i-2}(X_{8i-4}; \mathbf{Z}_2) \approx \mathbf{Z}_2 \text{ generated by } \boldsymbol{k}^{8i-2},$$

$$H^{8i-1}(X_{8i-3}; \mathbf{Z}_2) \approx \mathbf{Z}_2 \text{ generated by } \boldsymbol{k}^{8i-1}.$$

Proof of Theorem 6.1. The first assertion is trivial since $\pi_{2i+1}(B_{U(n)}) = 0$ for $i < n$. Consider the cohomology exact sequence

$$H^{2i+2}(X) \xrightarrow{\delta} H^{2i+3}(X_{2i+1}, X) \xrightarrow{a^*} H^{2i+3}(X_{2i+1}) \to H^{2i+3}(X)$$

with coefficients in $\pi_{2i+2}(X) \approx \mathbf{Z}$. Since $H^*(X) \approx \mathbf{Z}[c_1, \cdots, c_n]$, $\deg c_j = 2j$ (see [2], Theorem 21.3), we have $H^{2i+3}(X) = 0$ and $H^{2i+2}(X)$ is the direct

sum of $\pi(i+1)$ copies of $\mathbf{Z}$, where $\pi(i+1)$ is the number of partitions of $i+1$. By (5.2) and Lemma (1.1), δ is zero on the decomposable elements and $\delta c_{i+1} = \pm\, i! \cdot u$, where u is the fundamental class in $H^{2i+3}(X_{2i+1}, X) \approx \mathbf{Z}$. It follows that $H^{2i+3}(X_{2i+1})$ is cyclic of order $i!$, generated by $a^*u = \boldsymbol{k}^{2i+3}$.

Proof of Theorem 6.2. The first assertions are trivial since $\pi_{4j-2}(\boldsymbol{SO}(n))$, $\pi_{8i-4}(\boldsymbol{SO}(n))$ and $\pi_{8i-3}(\boldsymbol{SO}(n))$ are zero in the stable range (see [4]). Consider the cohomology exact sequence.

$$H^{4j}(X) \xrightarrow{\delta} H^{4j+1}(X_{4j-1}, X) \xrightarrow{a^*} H^{4j+1}(X_{4j-1}) \to H^{4j+1}(X) \xrightarrow{\delta'}$$

with coefficients in $\pi_{4j}(B_{SO(n)}) \approx \mathbf{Z}$. Since $H^*(B_{\boldsymbol{SO}}; \boldsymbol{R}) = \boldsymbol{R}[p_1, \cdots, p_r, \cdots]$, $H^*(B_{\boldsymbol{SO}(n)}; \mathbf{Z}_2) = \mathbf{Z}_2[w_2, \cdots, w_n]$, and $B_{\boldsymbol{SO}(n)}$ has no other torsion than 2-torsion (see [3], Appendix II), every class $c \in H^*(B_{\boldsymbol{SO}(n)})$ with $\deg c < n$ can be written in the form $c = P(p_1, \cdots, p_r) + \beta Q(w_2, \cdots, w_s)$, where β is the Bockstein homomorphism and P, Q are polynomials. Hence, $\delta\colon H^{4j}(X) \to H^{4j+1}(X_{4j-1}, X)$ kills every element except possibly p_j. It follows from (5.2) and Lemma (1.1) that $\delta p_j = \pm (2j-1)!\, a_j \cdot u$, where $u \in H^{4j+1}(X_{4j-1}, X)$ is the fundamental class. If $c \in H^{4j+1}(X)$, it has the form $\beta c'$, with $c' \in H^{4j}(X; \mathbf{Z}_2)$. Thus $\delta' c = 0$, except possibly if $c = \beta w_{4j} = W_{4j+1}$, and then $\delta' W_{4j+1} = \beta \delta w_{4j}$. Since by (5.2) δw_{4j} is the $4j$-dimensional Stiefel-Whitney class of the $\boldsymbol{SO}(n)$-bundle induced over S^{4j} by a map $S^{4j} \to B_{\boldsymbol{SO}(n)}$ representing a generator of $\pi_{4j}(B_{\boldsymbol{SO}(n)}) \approx \mathbf{Z}$, we obtain information about δw_{4j} from the following lemma:

LEMMA 6.4. *Let ξ be the stable $\boldsymbol{SO}(n)$-bundle induced over S^{4j} by a map representing a generator of $\pi_{4j}(B_{\boldsymbol{SO}(n)}) \approx \mathbf{Z}$. The Stiefel-Whitney class $w_{4j}(\xi) \in H^{4j}(S^{4j}; Z_2) \approx Z_2$ is different from zero if and only if S^{4j-1} is parallelizable.* (Compare Bott and Milnor [5].)

Since S^{4j-1} is parallelizable only for $j = 1, 2$ (see [5] or [6]), it follows that for $j \geqq 3$. the sequence

$$0 \to \mathbf{Z}_{(2j-1)!\,a_j} \xrightarrow{\tilde{a}^*} H^{4j+1}(X_{4j-1}) \to H^{4j+1}(X) \to 0,$$

is exact. In order to show that the sequence splits, consider the diagram

$$\begin{array}{ccccccc}
H^{4j}(X) & \xrightarrow{\delta} & H^{4j+1}(X_{4j-1}, X) & \xrightarrow{a^*} & H^{4j+1}(X_{4j-1}) & \to & H^{4j+1}(X) \\
\downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \phi_*} \\
H_2^{4j}(X) & \xrightarrow{\delta} & H_2^{4j+1}(X_{4j-1}, X) & \xrightarrow{a^*} & H_2^{4j+1}(X_{4j-1}) & \to & H_2^{4j+1}(X)
\end{array}$$

where $H_2{}^*$ is the cohomology with coefficients in $\mathbf{Z}_2$ and ϕ_* is induced by the coefficient epimorphism $\phi: \mathbf{Z} \to \mathbf{Z}_2$. Since $\phi_* u \neq 0$ and δ kills decomposable elements in $H_2{}^{4j}(X)$, it follows that $\phi_* \boldsymbol{k}^{4j+1} = 0$ if and only if $\delta w_{4j} \neq 0$. Hence, by Lemma 6.4 above, $\boldsymbol{k}^{4j+1}$ cannot be halved unless $j = 1$ or 2. Since $H^{4j+1}(X; \mathbf{Z})$ is the direct sum of copies of $\mathbf{Z}_2$, it follows that every element $\neq 0$ in $H^{4j+1}(X; \mathbf{Z})$ is the image of an element *of order two* in $H^{4j+1}(X_{4j-1}; \mathbf{Z})$. Thus (6.2′) splits for $j \geqq 3$.

Now, let $j = 1$ or 2. Since S^3 and S^7 are parallelizable, it follows that $\delta w_4 \neq 0$, $\delta w_8 \neq 0$. Thus $\boldsymbol{k}^5$ and $\boldsymbol{k}^9$ can be halved. Consider first the case $j = 1$: The sequence we are interested in reads

$$H^4(X) \to H^5(X_3, X) \xrightarrow{a^*} H^5(X_3) \to H^5(X) \xrightarrow{\delta'} \cdots.$$

Since $H^5(B_{SO(n)}) \approx \mathbf{Z}_2$ generated by W_5, the only possible value of the projection in $H^5(X)$ of $\frac{1}{2}\boldsymbol{k}^5$ is W_5. Therefore, δ' is trivial also in this case and exactness of (6.2′) holds for $j = 1$.

Let $j = 2$. Consider the sequence

$$H^8(X) \xrightarrow{\delta} H^9(X_7, X) \xrightarrow{a^*} H^9(X_7) \xrightarrow{p^*} H^9(X) \xrightarrow{\delta'} \cdots.$$

$H^9(B_{SO(n)}) \approx \mathbf{Z}_2 + \mathbf{Z}_2 + \mathbf{Z}_2$ generated by $p_1 W_5$, $(W_3)^3$ and W_9. Let h be an element in $H^9(X_7)$ such that $2h = \boldsymbol{k}^9$. Claim: $p^*h = W_9 + \alpha p_1 W_5 + \beta (W_3)^3$, where α and β are remainders mod 2 depending on the choice of h. This is equivalent to proving $p^*h \neq \alpha p_1 W_5 + \beta (W_3)^3$. Since p^* is an isomorphism in dimensions < 8, there exist classes p'_1, W'_5, W'_3 whose projection under p^* are p_1, W_5, W_3. Thus $\alpha p'_1 W'_5 + \beta (W'_3)^3 \in H^9(X_7)$ is an element of order 2 whose image by p^* is $\alpha p_1 W_5 + \beta (W_3)^3$. If p^*h were equal to $\alpha p_1 W_5 + \beta (W_3)^3$, we would get an element $h' = h - \alpha p'_1 W'_5 - \beta (W'_3)^3$ with the properties $p^*h' = 0$ and $2h' = \boldsymbol{k}^9$. Such an element however does not exist. It follows from $p^*h = W_9 +$ decomposable elements, that $\delta W_9 = 0$. Hence (6.2′) is seen to be exact in any case.

To obtain the exactness of (6.2″), consider the cohomology sequence

$$H^{8i+s-1}(X) \xrightarrow{\delta} H^{8i+s}(X_{8i+s-2}, X) \xrightarrow{a^*} H^{8i+s}(X_{8i+s-2}) \xrightarrow{p^*} H^{8i+s}(X) \xrightarrow{\delta'}$$

with coefficients in $\pi_{8i+s-2}(SO(n)) \approx \mathbf{Z}_2$ for $s = 2$ or 3.

We have to prove that δ and δ' are zero. Since δ and δ' kill decomposable elements, it suffices to prove $\delta w_{8i+1} = 0$, $\delta w_{8i+2} = 0$, $\delta' w_{8i+2} = 0$, $\delta' w_{8i+3} = 0$. The first two assertions follow from (5.2), by which

$$\delta w_{8i+s-1} \in H^{8i+s}(X_{8i+s-2}, X; \mathbf{Z}_2) \approx \mathbf{Z}_2$$

is the value $\alpha^* w_{8i+s-1}[S^{8i+s-1}]$ of the $(8i+s-1)$-th Stiefel-Whitney class of the $\boldsymbol{SO}(n)$-bundle over S^{8i+s-1} induced by a map representing the generator α of $\pi_{8i+s-1}(B_{\boldsymbol{SO}(n)}) \approx \boldsymbol{Z}_2$. Since such a Stiefel-Whitney class vanishes, it follows that $\delta w_{8i+s-1} = 0$. To prove $\delta' w_{8i+s} = 0$, observe that

$$w_{8i+2} = Sq^2 w_{8i} \text{ for } i \geqq 1, \text{ and } w_{8i+3} = Sq^1 w_{8i+2}$$

(see Wu Wen-Tsün [13] and Borel [1]). It follows that

$$\delta' w_{8i+2} = \delta' Sq^2 w_{8i} = Sq^2 \delta w_{8i} = 0, \text{ and similarly } \delta' w_{8i+3} = 0.$$

The splitting of (6.2″) is trivial since every element in $H^{8i+s}(X_{8i+s-2}; \boldsymbol{Z}_2)$ has order two.

It remains to prove Lemma 6.4. Let ξ' be the associated bundle with fibre $V_{n,n-4j+1} = \boldsymbol{SO}(n)/\boldsymbol{SO}(4j-1)$, and let

$$p_*: H^{4j}(S^{4j}; \pi_{4j-1}(\boldsymbol{SO}(n))) \to H^{4j}(S^{4j}; \pi_{4j-1}(V_{n,n-4j+1}))$$

be induced by $p: \pi_{4j-1}(\boldsymbol{SO}(n)) \to \pi_{4j-1}(V_{n,n-4j+1})$. Identifying $H^{4j}(S^{4j}; \pi_{4j-1}(\boldsymbol{SO}(n)))$ with $\pi_{4j-1}(\boldsymbol{SO}(n)))$, we clearly have $p_* \alpha = w_{4j}(\xi)$, where α is a generator of $\pi_{4j-1}(\boldsymbol{SO}(n))$. Since $\pi_{4j-1}(V_{n,n-4j+1}) \approx \boldsymbol{Z}_2$, it follows that $w_{4j}(\xi)$ is different from zero if and only if $p: \pi_{4j-1}(\boldsymbol{SO}(n)) \to \pi_{4j-1}(V_{n,n-4j+1})$ is surjective. The commutative diagram

$$\begin{array}{ccc} \pi_{4j-1}(\boldsymbol{SO}(4j)) & \to & \pi_{4j-1}(\boldsymbol{SO}(n)) \\ \downarrow p' & & \downarrow p \\ \pi_{4j-1}(S^{4j-1}) & \to & \pi_{4j-1}(V_{n,n-4j+1}) \end{array}$$

shows that p is surjectice if and only if p' is; in other words, if and only if S^{4j-1} is parallelizable.

Remark. The proof in [6] can be improved by using the above diagram from which Lemma 2 of [6] follows immediately.

Proof of Theorem 6.3. Let $X = B_{\boldsymbol{Sp}(n)}$. Again the first statements of the theorem follow trivially from the results of [4].

Consider the exact sequence

$$H^{4j}(X) \xrightarrow{\delta} H^{4j+1}(X_{4j-1}, X) \xrightarrow{a^*} H^{4j+1}(X_{4j-1}) \to H^{4j+1}(X)$$

with integer coefficients ($\pi_{4j}(X) \approx \boldsymbol{Z}$ by [4]).

Since $H^*(B_{\boldsymbol{Sp}(n)}; \boldsymbol{Z}) = \boldsymbol{Z}[K_1, \cdots, K_n]$, $\deg K_j = 4j$, it follows that

$H^{4j+1}(X) = 0$, and the multiples of K_j are the only non-decomposable elements of $H^{4j}(X)$. By (5.2) and Lemma 1.1,

$$\delta K_j = (2j-1)!\, b_j \cdot u,$$

where u is the fundamental class in $H^{4j+1}(X_{4j-1}, X; \mathbf{Z}) \approx \mathbf{Z}$.

It follows that $H^{4j+1}(X_{4j-1})$ is cyclic of order $(2j-1)!\, b_j$ generated by $\boldsymbol{k}^{4j+1}$. The assertions about $\boldsymbol{k}^{8i-1}$ and $\boldsymbol{k}^{8i-2}$ are trivial.

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NORMALITY IN SUBSETS OF PRODUCT SPACES.*

By H. H. Corson.

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1. Introduction. This is a study of the topological properties of some of the subsets of what will be called a Σ-product. For the purposes of this introduction it will be enough to define a Σ-product of copies R_a of the real line where a is in an uncountable index set A. This is the subset Σ of the full product of the R_a which consists of those points which have at most a countable number of non-zero coordinates. The main properties of these spaces are, loosely speaking, strong types of normality and weak types of separability. For instance, it is proved that Σ is not only normal but that the collection of all the neighborhoods of the diagonal in $\Sigma \times \Sigma$ forms a uniformity for Σ. The space Σ is not paracompact or separable although each metric space which is the continuous image of Σ, or is a closed subset of Σ, is separable. Moreover, Σ has a dense subset which is the union of a countable number of compact spaces.

As applications of these theorems, two counterexamples are constructed. First, a normal space is constructed whose real-compactification is not normal. (See Section 3 for the definition of real-compactification.) This question has been asked in a preliminary version of a book by L. Gilman and M. Jerison, and has also been considered elsewhere. Second, it is shown that a space F_0 which was first defined by A. H. Stone (see Section 4) has the property that the collection of all the neighborhoods of the diagonal in $F_0 \times F_0$ forms a complete uniformity for F_0, but F_0 is not paracompact. This answers in the negative a conjecture of Kelley's [9, pages 208-209]. (It is stated that a counterexample had been constructed, but an error was found during a seminar at Purdue.)

Σ is also proved to have the following additional properties. It is a $C(X)$ for a Lindelöf space X, where $C(X)$ is the collection of continuous real functions on X under either the simple topology or the compact open topology—these topologies being the same for this X. Every metric space can be imbedded as a subspace of some Σ-product of real lines. However, it is shown that Σ has a non-normal subset.

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Perhaps it is best to mention that in the literature there are a few cases of results on Σ-products of compact spaces X_a, $a \in A$. These are defined by specifying that one point in each X_a is to be 0, and then imitating the above definition of the Σ-product of real lines. These results also follow from the theorems in this paper. However, the reader will see that, for the applications which were described above, it is essential that one consider a Σ-product of non-compact spaces. For the theorems that remain true, most of the proofs given here can be simplified if X_a is compact for each a.

Certain conventions will be followed throughout. A topological space will be a completely regular, Hausdorff space. For a subset F of a topological space X, F^- will be the closure of F in X, while F' will be the complement of F in X.

Finally, some comments of J. R. Isbell are gratefully acknowledged.

2. Normality of Σ-products. Although a Σ-product of real lines was defined in the introduction, it will be possibly usually to deal with a larger class of spaces. These are defined as follows.

Definition. For a collection $\{X_a : a \in A\}$ of topological spaces, let $P = P\{X_a : a \in A\}$ be the topological product of the X_a. Then a *Σ-product* of the X_a is a subset Σ of P with the property that there exists $p = (p_a) \in P$ such that $q = (q_a) \in \Sigma$ if and only if $q_a \neq p_a$ for at most a countable number of $a \in A$. Such a point p is called the *base point* of this Σ-product.

Perhaps it is best to point out immediately that one Σ-product of the X_a may be different from another if the base point is changed. For instance, suppose that each X_a is a copy of the unit interval $[0, 1]_a$ plus an isolated point ∞_a, and let A be uncountable. Then the Σ-product with $p = (\infty_a)$ as the base point is not homeomorphic to that with $p' = (0_a)$ as the base point. In fact, the first space has a point (the base point) whose component is just a single point, while the second Σ-product has no such point.

It will now be proved that a Σ-product of complete, separable metric spaces is normal. The following lemma will be used in the proof, but first some terminology is needed.

Let $\{H_s : s \in S\}$ be a collection of subsets of a topological space X, and let U be an open subset of X. It will be said that $\{H_s : s \in S\}$ can be separated in U if there is a disjoint collection of open sets $\{U_s : s \in S\}$ such that $U \cap H_s \subset U_s$ for all $s \in S$.

LEMMA 1. *Let H_1 and H_2 be two subsets of the product of a topological space Y with a metric space M, and suppose $\mathcal{U}$ is an open cover of M. If H_1*

and H_2 can be separated in each $Y \times U$, $U \in \mathcal{U}$, then H_1 and H_2 can be separated in $Y \times M$.

Proof. A. H. Stone has proved in [12] that each metric space is paracompact. That is, for each open cover $\mathcal{U}$ of M there is a locally finite refinement $\mathcal{V}$ of $\mathcal{U}$. More explicitly, $\mathcal{V}$ has the property that each $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$, and each $x \in M$ has a neighborhood that meets only a finite number of $V \in \mathcal{V}$. Since M is normal, one may even choose V such that the closure of each $V \in \mathcal{V}$ is contained in some $U \in \mathcal{V}$. Let $\mathcal{V}$ be such a refinement for the $\mathcal{U}$ in the statement of Lemma 1.

For each $V \in \mathcal{V}$ there are open sets V_1 and V_2 in $Y \times V$ such that $H_i \cap (Y \times V) \subset V_i$, $i = 1$ or 2, and $V_1 \cap V_2 = \phi$. This is true since H_1 and H_2 can be separated in each $Y \times U$, and hence in $Y \times V$. Since $V^- \subset U$ for some $U \in \mathcal{U}$, one may even assume that $V_1^- \cap H_2 = V_2^- \cap H_1 = \phi$. Let $S_i = \cup\{V_i : V \in \mathcal{V}\}$, $i = 1, 2$. Define $T_1 = S_1 \cap S_2^{-\prime}$, and $T_2 = S_2 \cap S_1^{-\prime}$. Since T_1 and T_2 are open and disjoint, all that is left to prove is that $H_1 \subset T_1$ and $H_2 \subset T_2$.

Let $h \in H_1$, then $h = (y, m)$, where $y \in Y$ and $m \in M$. Choose a neighborhood N of m such that N meets only a finite number of $V \in \mathcal{V}$. Then $Y \times N$ is a neighborhood of h which meets only a finite number of $Y \times V$, and hence only a finite number of V_2. Let $\mathcal{W}$ be this finite collection of V_2. Recall that $V_2^- \cap H_1 = \phi$ for each $V \in \mathcal{V}$. Hence $(\cup\mathcal{W})' \cap (Y \times N)$ is a neighborhood of h which meets none of the V_2. It follows that $H_1 \subset T_1$, and a similar argument shows that $H_2 \subset T_2$.

Remark. Lemma 1 is true for paracompact M, as the proof shows, or more generally for a normal space M and a locally finite cover $\mathcal{U}$ of M. Moreover, if a discrete collection of closed sets $\{H_s : s \in S\}$ is substituted for H_1, H_2, Lemma 1 is still true. Almost the same proof may be used to show this.

THEOREM 1. *A Σ-product of complete metric spaces is normal.*

Proof. Let $\{M_a : a \in A\}$ be a collection of complete metric spaces with ρ_a being a metric on M_a for which M_a is complete. For each finite subset F of A, let $P_F = P\{M_a : a \in F\}$. The metric ρ_F on a subspace of such a P_F will always be taken to be $\rho_F(x, y) = \sup\{\rho_a(x_a, y_a) : a \in F\}$. Henceforth the subscripts on the ρ will be omitted. Also, denote the Σ-product of the M_a by Σ, where the base point of Σ is $p = (p_a)$.

Let H_1 and H_2 be subsets of Σ which cannot be separated in Σ. It will be shown that there are sequences $\{h_i\}$ and $\{k_i\}$ with $h_i \in H_1$, $k_i \in H_2$, and

with $\{h_i\}$ and $\{k_i\}$ converging to the same point in Σ. Hence H_1 and H_2 cannot be closed, disjoint subsets of Σ, and the theorem will follow.

Obviously H_1 and H_2 are not empty since they cannot be separated. Let x_1 be any point in H_1; then there is a countable set of indices C_1 such that $(x_1)_a = p_a$ if $a \notin C_1$. Arrange C_1 in a simple sequence. F_1 will be a finite set of indices which includes the first member of C_1. Put $S_1 = \Sigma$.

Suppose n quadruples (x_i, C_i, F_i, S_i) have been chosen with the following properties: $x_i \in H_1$ for odd i, and $x_i \in H_2$ for even i. Each C_i is a countable set of indices which includes all $a \in A$ such that $(x_i)_a \neq p_a$. Each C_i is arranged in a simple sequence. F_i is a finite set of indices which contains $F_1 \cup \cdots \cup F_{i-1}$ as well as the first i elements of each C_j, $j \leqq i$. Each S_i for $2 \leqq i \leqq n$ is contained in S_j for $j \leqq i$, and S_i satisfies the following condition. If π_i is the natural projection of Σ onto $P_{F_{i-1}}$, then there is a sphere T_i of radius $1/i$ in $P_{F_{i-1}}$ such that $S_i = \pi_{i-1}^{-1}(T_i)$. Moreover, assume that H_1 and H_2 cannot be separated in S_i, and that $x_i \in S_i$. The quadruple for $n+1$ can be chosen as follows.

Let $\mathcal{U}$ be the covering of $M = \pi_n(S_n)$ by spheres of radius $1/(n+1)$. It is easily seen that S_n is homeomorphic to the topological product of M with some topological space, and that M is a metric space. Hence by Lemma 1 there is a sphere T_{n+1} in $\mathcal{U}$ such that H_1 and H_2 cannot be separated in $S_{n+1} = \pi_n^{-1}(T_{n+1})$, because H_1 and H_2 cannot be separated in S_n. In particular, $H_1 \cap S_{n+1}$ and $H_2 \cap S_{n+1}$ are not empty, so one may choose $x_{n+1} \in H_1$ if $n+1$ is odd or $x_{n+1} \in H_2$ if $n+1$ is even. C_{n+1} and F_{n+1} are chosen in the obvious way.

Let $\{h_i\} = \{x_j : j \text{ is odd}\}$, and $\{k_i\} = \{x_j : j \text{ is even}\}$; it will be shown that $\{h_i\}$ and $\{k_i\}$ converge to the same point in Σ. However this is equivalent to showing that $\{x_i\}$ converges. In order to establish this last statement, let a be an element of A, but a not in any C_i. In this case $(x_i)_a = p_a$ for all i. If $a \in C_i$ for some i, then $a \in F_n$ for all n sufficiently large. By the definition of the metric on P_{F_n} and by the choice of x_i, $\rho((x_i)_a, (x_j)_a) < 1/n$ for all n sufficiently large and for all i and j greater than n. Hence $\{(x_i)_a\}$ is a Cauchy sequence in M_a, and it follows that $\{x_i\}$ converges to some $x_0 \in P$, the product of the M_a. But $x_0 \in \Sigma$ since $C = \cup\{C_i : i = 1, 2, \cdots\}$ is countable, and the proof is complete.

COROLLARY 1. *The product of a countable number of Σ-products of complete metric spaces is normal and countably paracompact.*

Proof. Let X denote this product, and let I be the unit interval. By [5] it is sufficient to prove that $X \times I$ is normal. However, it is obvious that $X \times I$ is simply another Σ-product of complete metric spaces.

Remark. E. Michael has pointed out to me that the proof of Theorem 1 may be used, almost as it stands, to show that Σ is collectionwise normal. That is, a discrete collection of closed subsets of Σ may be separated in Σ. The remark following Lemma 1 was inserted for this reason. Some of Michael's observations have been incorporated into the proofs of Lemma 1 and Theorem 1 so that the statement made above would be easy to verify. However, if each of the metric spaces is also separable, it is proved in Section 4 that Σ has even stronger properties.

3. The real-compactification of a Σ-product. First, a definition and some results given in [8] will be reviewed. "Real-compact" is the same as "Q" in the sense of Hewitt. Specifically, X is *real-compact* if it has the property that, whenever there is a topological space Y and a homeomorphism f of X into a dense subset of Y such that every continuous real function on $f(X)$ can be extended to Y, then $f(X) = Y$. It is proved in [8] that each separable metric space is real-compact, and that a product of real-compact spaces is real-compact. It is also shown that each topological space X can be imbedded as a dense subspace of a real-compact space vX in such a way that each continuous real function on X can be extended to vX. Moreover, vX is unique up to a homeomorphism, and vX is called the *real-compactification* of X.

In the present section, $v\Sigma$ is determined for some Σ-products.

THEOREM 2. *If P is the product of separable metric spaces $\{M_a : a \in A\}$, and if Σ is a Σ-product of the M_a, then $P = v\Sigma$.*

Proof. By the above remarks P is real compact. Σ is dense in P. Hence, by the uniqueness of $v\Sigma$, it suffices to prove that every continuous real function on Σ can be extended to P.

If C is a subset of A, let π_C be the natural projection from P to $P_C = P\{M_a : a \in C\}$. Speaking loosely, Mazur has proved in [10] that if f is any continuous real function on Σ, then there is a countable set C contained in A and a continuous real function g on P_C such that $f = g\pi_C$. In fact, Σ is a subset of P which is invariant under projection in Mazur's terminology, and all such subsets of P are shown by him to have the above property. It is obvious from this that every continuous real function on Σ can be extended to P.

However, Mazur proves more than is asserted above. He is forced for this reason to assume that, if $\aleph$ is the cardinality of A, then $\aleph$ is less than the first inaccessible cardinal. (See [10] for the definition of this.) This

restriction on A may be avoided here as follows. Bockstein proved in [2] that, given any two disjoint open set U_1 and U_2 in P, there is a countable subset D of A and disjoint open sets V_1 and V_2 in P_D such that $\pi_D(U_i) \subset V_i$, $i = 1, 2$. His proof may be used to demonstrate the same statement for Σ. Since every continuous real function on Σ is determined by a countable number of pairs of disjoint open sets in Σ, it follows that $f = g\pi_C$ for some countable $C \subset A$. This completes the proof of Theorem 2.

COROLLARY 2. *There is a normal space Σ whose real compactification $v\Sigma$ is not normal.*

Proof. Let Σ be as in Theorem 2 with A uncountable and with each M_a complete but not compact. Then by Theorem 1, Σ is normal. By Theorem 2, $P = v\Sigma$. By a result of A. H. Stone's, P is not normal [12]. (A proof that P is not normal is also given in Section 4.)

4. The neighborhoods of the diagonal in $\Sigma \times \Sigma$. Section 2 contains all the positive results that I know on Σ-products of spaces which are complete metric but not necessarily separable. If each of the factors is separable, however, we have the following theorem.

THEOREM 3. *Let Σ be a Σ-product of complete separable metric spaces $\{M_a : a \in A\}$. Then the family of neighborhoods of the diagonal in $\Sigma \times \Sigma$ is a uniform structure for Σ.*

Proof. Let Δ be the diagonal in $\Sigma \times \Sigma$ and U be a fixed neighborhood of Δ. It is sufficient to find a metric space M and a function ϕ from Σ onto M such that $\{(x, y) : \rho(\phi(x), \phi(y)) < 1\} \subset U$. ($\rho$ is the metric in M.)

Some preliminary construction is needed. First observe that Corollary 1 implies that $\Sigma \times \Sigma$ is normal. Δ being closed in $\Sigma \times \Sigma$, there is a continuous function f from $\Sigma \times \Sigma$ to the unit interval such that $f(\Delta) = O$ and $f(U') = 1$. It is easy to see that $\Sigma \times \Sigma$ is a Σ-product of $\{M_a \times N_a : a \in A\}$, where M_a is homeomorphic to N_a for all $a \in A$. In the proof of Theorem 2 it has been shown that there is a countable subset $C \subset A$ such that $f = g\pi_C$, where π_C is the projection onto $P_C = P\{M_a \times N_a : a \in C\}$ and g is a continuous real function on P_C. Let $T_C = P\{M_a : a \in C\}$. Clearly, $T_C \times T_C$ may be identified with P_C. Denote by Δ_C the diagonal in $T_C \times T_C$. One see that $\Delta_C = \pi_C(\Delta)$. Let $U_C = \{(x, y) \in T_C \times T_C : g(x, y) < 1\}$.

The existence of ϕ and M can now be proved as follows. All the neighborhoods of Δ_C in $T_C \times T_C$ do form a uniformity for T_C, since T_C is metric and hence paracompact [9, Theorem 5.27]. It follows that there is

a metric space M and a continuous function ϕ_C from T_C to M such that, in $T_C \times T_C$, $\{(x,y): \rho(\phi_C(x), \phi_C(y)) < 1\} \subset U_C$. Define ϕ to be $\phi_C \sigma_C$, where σ_C is the projection from Σ to T_C.

It will be proved that ϕ has the required property. Let $(x,y) \in \Sigma \times \Sigma$ such that $\rho(\phi(x), \phi(y)) < 1$. Then $\rho(\phi_C(\sigma_C(x)), \phi_C(\sigma_C(y))) < 1$, so $(\sigma_C(x), \sigma_C(y)) \in U_C$ by selection of ϕ_C. Hence $f(x,y) = g\pi_C(x,y) < 1$ by the choice of U_C, and it follows that $(x,y) \in U$ since f is equal to 1 on U'.

COROLLARY 3. *A Σ-product of complete separable metric spaces is collectionwise normal. (See remarks at the end of Section 2 for the definition of collectionwise normality.)*

Proof. It is proved in [3] that collectionwise normality is implied by the property that all the neighborhoods of the diagonal is a uniformity.

The proof of Theorem 3 implies the following additional fact which will be needed.

COROLLARY 4. *If X is a metric space and also a continuous image of a Σ-product of complete separable metric spaces, then X is separable.*

Proof. X is paracompact, since it is metric. Hence for each open cover $\mathcal{V}$ of X there is a neighborhood U of the diagonal in $X \times X$ such that $\{U(x): x \in X\}$ refines $\mathcal{V}$ [9; Theorem 5.28]. If f is a continuous function which maps Σ onto X, then $U_0 = (f \times f)^{-1}(U)$ is a neighborhood of the diagonal in $\Sigma \times \Sigma$. By the proof of Theorem 3 there is a function ϕ from Σ to a metric space M such that $\{(x,y): \rho(\phi(x), \phi(y)) < 1\} \subset U_0$ However, this M is a separable metric space since it is the image of the separable space T_C. Hence there is a countable subset $\{y_i\}$ of Σ such that $\{U_0(y_i): i = 1, 2, \cdots\}$ covers Σ. It follows that $\{U(f(y_i)): i = 1, 2, \cdots\}$ covers X, and hence $\mathcal{V}$ has a countable subcover.

Note. There is another proof of Corollary 4 which shows that it remains true for any dense subset of a full product of separable metric spaces.

It is interesting to note that Bockstein's theorem can be used to give a new proof of the theorem [12] that P, the product of an uncountable number of non-compact metric spaces, is not normal. The first part of Stone's proof goes as follows. First note that P has a closed subset P_0 which is homeomorphic to the product of an uncountable number of copies of the integers. Hence it is enough to show that $P_0 = P\{J_a: a \in A\}$ is not normal, where J_a is a copy of the integers. Stone then defines two closed subsets F_0 and F_1 of P_0, where F_i is the set of p in P_0 such that, for any given integer $n \neq i$,

$i=0$ or 1, $p_a=n$ for at most one $a\in A$. It is possible to show very quickly that F_0 and F_1 cannot be separated in P_0. In fact, if F_0 and F_1 could be separated, then there would be a countable subset $C\subset A$ such that $\pi_C(F_0)$ and $\pi_C(F_1)$ could be separated in $P_C=P\{J_a: a\in C\}$, because of Bockstein's theorem [2]. However, given a $C=\{a_j: j=1,2,\cdots\}$, the point $p=(p_a)\in P_C$, where $p_{a_j}=j$, is in $\pi_C(F_0)\cap\pi_C(F_1)$.

On the other hand, F_0 as a topological space has some interesting properties itself. In fact, it is a counterexample to the conjecture of Kelley's mentioned in the introduction.

THEOREM 4. *F_0 has the property that the collection $\mathcal{W}$ of all the neighborhoods of the diagonal in $F_0\times F_0$ forms a complete uniformity for F_0. However, F_0 is not paracompact.*

Proof. Using the notation introduced above, F_0 is a closed subset of P_0, and hence of Σ, the Σ-product of the J_a. Since Σ has the property that the collection of all the neighborhoods of the diagonal forms a uniformity, so does each closed subset. (More generally, it is easy to prove that this property is inherited by closed subsets.) Moreover, F_0 is closed in P_0, and P_0 is complete in the product uniform structure. Hence F_0 is complete in some uniform structure. Consequently F_0 is complete in its strongest uniform structure $\mathcal{W}$ [9, problem 6.L(a)].

In order to show that F_0 is not paracompact, notice the following facts. Since Σ is collectionwise normal by Corollary 3, then each continuous function from the closed subset F_0 of Σ to a Banach space extends to Σ [6]. By Corollary 4, every continuous image M of Σ, where M is metrizable, is separable. Thus the same is true of F_0, since it is well known that each metric space can be imbedded in a Banach space. If F_0 were paracompact, then for each open cover $\mathcal{V}$ there would be a neighborhood U of the diagonal in $F_0\times F_0$ such that $\{U(x): x\in F_0\}$ refines $\mathcal{V}$ [9; Theorem 5.28]. But each such U is a member of the strongest uniformity for F_0. Hence there is a metric space M which is the continuous image of F_0 under ϕ, and such that $\{(x,y): \rho(\phi(x),\phi(y))<1\}\subset U$. Since M must be separable, $\mathcal{V}$ has a countable subcover. Hence it follows that if F_0 were paracompact, then it would be Lindelöf. However, consider the open cover of Σ by sets of the form $V_a=\{p\in\Sigma: p_a=0\}$. Then $\mathcal{V}=\{V_a: a\in A\}$ covers Σ and hence F_0. If $\{V_{a_i}: i=1,2,\cdots\}$ is a countable subset of $\mathcal{V}$, then the point $p=(p_a)\in F_0$ is not in any V_{a_i}, $i=1,2,\cdots$, where $p_a=0$, if $a\neq a_i$ and $p_{a_i}=i$. Therefore $\mathcal{V}$ has no countable subcover, F_0 is not Lindelöf, and hence F_0 is not paracompact.

Remark. Since F_0 is closed in Σ, Corollary 1 implies that F_0 is also countably paracompact.

There is another corollary to Theorem 3, which is a known result. If Σ is a non-trivial Σ-product and $\lambda\Sigma$ is any compactification of Σ, then $\Sigma \times \lambda\Sigma$ is not normal. It suffices to prove this in the case where each factor of Σ is a space with exactly two points. However, since the collection of all the neighborhoods of the diagonal is a uniformity for Σ, if $\Sigma \times \lambda\Sigma$ were normal then Σ would be paracompact [4]. Almost the same proof which was used to prove that F_0 is not paracompact may be applied to show that Σ is not paracompact.

It follows from the above remark that a Σ-product of compact spaces need not be normal. However, it is known and easily proved that such Σ are countably compact in this case. As for metric spaces which are not complete, I do not know if a Σ-product of copies of the rational numbers is normal.

5. Metrizable subsets of Σ-products. Σ-products are ordinarily not metrizable or even paracompact, as was mentioned in the introduction. However, the following is true.

PROPOSITION 1. *Every metric space can be imbedded as a subspace of a Σ-product of copies of the unit interval.*

Proof. If M is a metric space, it is shown in [1] that there are a countable number of collections $\mathcal{A}_i$ of open sets such that $\cup\{\mathcal{A}_i : i = 1, 2, \cdots\}$ is a basis for M and such that each $\mathcal{A}_i$ is a discrete collection. (The latter means that each point of M is contained in an open set which meets at most one member of $\mathcal{A}_i$.) For each i and each A in $\mathcal{A}_i$, one may choose a countable collection of open sets B_n such that $B_n^- \subset A$ and $A = \cup\{B_n : n = 1, 2, \cdots\}$.

For each B let f_B be a continuous function from M to the unit interval such that $f_B(m) = 1$ if $m \in B$ and $f_B(m) = 0$ if m is not in the A which corresponds to B. For each B let I_B be a copy of the unit interval, and let Σ be the Σ product of the I_B, where the base point is the point which has every coordinate equal to zero. Let ϕ be the function from M to X defined by $(\phi(m))_B = f_B(m)$. It follows from [9, Lemma 4.5] that ϕ is a homeomorphism, which completes the proof.

Remark. On the other hand, if M is a metric space which is a *closed* subspace of a Σ-product of unit intervals, then M must be separable. This follows from the results in Section 4.

6. Non-normal subspaces of Σ-products. Although it has been shown that Σ-products have many normal subspaces, and although more normal subspaces will be exhibited presently, it is nevertheless true that every non-trivial Σ-product has a subset which is not normal. This is easily demonstrated using the next two lemmas.

LEMMA 2. *If X is normal and if A is a subspace of X such that each continuous real function on X is bounded on A, then every sequence in A has a cluster point.*

Lemma 2 is known and easily proved. Lemma 3 is due to Melvin Henriksen [7].

LEMMA 3. *Let A be a subspace of X such that every continuous real function on A can be extended to X. Then for $x \in A$, every continuous real function on $A - \{x\}$ can be extended to $X - \{x\}$.*

PROPOSITION 2. *Let Σ be a Σ-product of uncountably many spaces, each having more than one point. Then Σ has a non-normal subset.*

Proof. By hypothesis, each factor X_a of Σ has a two-point subset Y_a. Assume that if $p = (p_a)$ is the base point of Σ then $p_a \in Y_a$. It will be proved that Σ_0 has a non-normal subset, where Σ_0 is the Σ-product of the Y_a with p as base point. Since $\Sigma_0 \subset \Sigma$, Proposition 2 will follow.

Let P_0 be the full product of the Y_a, then $P_0 = v\Sigma_0$ by Theorem 2. Choose some point $q \in P_0$ which is not in Σ_0. Since every continuous real function on Σ_0 can be extended uniquely to P_0, $P_0 - \{q\}$ has the same property with respect to P_0. That is, $P_0 = v(P_0 - \{q\})$. But P_0 is compact, therefore every continuous function on $P_0 - \{q\}$ is bounded. Also, P_0 may be considered to be a topological group and is consequently homogeneous. Hence every continuous function on $P_0 - \{x\}$ is bounded for all x in P_0.

Let x_0 be a fixed point of Σ_0. Lemma 3 implies that each continuous real function on $\Sigma_0 - \{x_0\}$ can be extended to $P_0 - \{x!\}$. Hence all continuous functions on $\Sigma_0 - \{x_0\}$ are bounded, since $P_0 - \{x_0\}$ has been shown to have this property. However, $\Sigma_0 - \{x_0\}$ contains a sequence with no cluster point. To define this sequence, let x_i be a point in $\Sigma - \{x_0\}$ such that $(x_i)_a \neq (x_0)_a$ for exactly one a. A countable number of distinct a_i has no cluster point. By Lemma 2, $\Sigma_0 - \{x_0\}$ is not normal.

7. A dense Lindelöf subspace of Σ. Suppose Σ is a Σ-product of $\{M_a : a \in A\}$ with base point $p = (p_a)$. Then a *σ-product* of the M_a is the set of $q = (q_a) \in \Sigma$ such that $q_a \neq p_a$ for at most a finite number of a. If σ

is such a σ-product, then $\sigma = \cup\{\sigma_i : i = 0, 1, 2, \cdots\}$, where σ_i is the set of $q = (q_a) \in \sigma$ such that $q_a \neq p_a$ for at most i of the $a \in A$.

LEMMA 4. *If M and each M_a are separable metric, then $\sigma_i \times M$ is Lindelöf.*

Proof. The proof is by induction. It is trivial for $i = 0$, so suppose that the statement is true for σ_n. Let $\mathcal{O}$ be an open cover of $\sigma_{n+1} \times M$. In order to show that there is a countable subcover of $\mathcal{O}$, it may be assumed that each member of $\mathcal{O}$ is a member of the usual product basis for open sets in $\sigma_{n+1} \times M$. Since M is separable metric there is a countable subset of $\mathcal{O}$, say $\{O_i : i = 1, 2, \cdots\}$, which covers $p \times M$, where p is the base point. Since each O_i is a member of the basis, there is a finite subset F_i of A, a neighborhood V_a^i of p_a in M_a for each $a \in F_i$, and V^i open in M such that

$$O_i = V^i \times P\{V_a^i : a \in F_i\} \times P\{M_a : a \notin F_i\}.$$

Let $G_b^i = V^i \times (V_b^i)' \times P\{M_a : a \neq b\}$ for each $b \in F_i$; then G_b^i is homeomorphic to $V \times (V_b^i)' \times \sigma_n$. Since $(V_a^i)' \times V^i$ is separable metric, G_b^i is covered by a countable number of elements of $\mathcal{O}$. However,

$$\sigma_{n+1} \times M = (\cup\{O_i : i = 1, 2, \cdots\}) \cup (\cup\{G_b^i : b \in F_i, i = 1, 2, \cdots\}).$$

Hence $\mathcal{O}$ has a countable subcover.

Lemma 4 is related to a problem of Michael's [11]. He asks if the product of a paracompact space and a metric space is paracompact, but it is even known of the product of a Lindelöf space and a separable metric space is paracompact.

PROPOSITION 3. *The σ-product of separable metric spaces is Lindelöf. Hence each Σ-product of separable metric spaces has a dense Lindelöf subspace.*

Proof. $\sigma = \cup\{\sigma_i : i = 1, 2, \cdots\}$. By Lemma 4, $\sigma_i \times M$ is Lindelöf if M has only one point. Hence σ_i is Lindelöf, and the union of a countable number of Lindelöf spaces is Lindelöf.

8. Σ-products of real lines.

PROPOSITION 4. *Let M_a be homeomorphic to the real line for each $a \in A$. Then σ, the σ-product of the M_a, is the union of a countable number of compact sets*

Proof. Define $(\sigma_i)_j$ to be the set of $p = (p_a) \in \sigma_i$ such that $|p_a| \leqq j$ for each $a \in A$. It will suffice to show that $(\sigma_i)_j$ is compact for each i and j. However, it is easy to see that $(\sigma_i)_j$ is closed in the full product $P\{N_a : a \in A\}$,

where N_a is the closed interval $[-j, j]$ contained in M_a. Hence $(\sigma_i)_j$ is compact.

PROPOSITION 5. *Let Σ be a Σ-product of copies of the real line. Then Σ is homeomorphic to $C(X)$, all the continuous real function on a Lindelöf space X, where $C(X)$ has either the simple topology or the compact open topology—these topologies being the same for this X.*

Proof. Let the points in X be the index set A together with a point called ∞. A basis for the open sets in X is defined as follows. Each point of A is open, and complements (in X) of countable subsets of A are open. It is easy to see that Σ is homeomorphic to $C_0(X)$, the functions in $C(X)$ which vanish at ∞. However, $C(X)$ is homeomorphic to $C_0(X) \times R$, where R is the real line, which is in turn homeomorphic to $\Sigma \times R$. If A is infinite, then $\Sigma \times R$ is homeomorphic to Σ, while the proposition is obvious if A is finite.

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ERRATA TO OUR PAPER "LOCAL UNIQUENESS, ETC."*

(this Journal, vol. 80, pp. 421-430.)

By FRED BRAUER and SHLOMO STERNBERG.

A number of errors occur in our paper which we should like to rectify. In Theorem 1 we give a uniqueness criterion which we state is sufficiently general so as not to imply the convergence of successive approximations. We show this in Section 4 by giving the Müller example as an instance of a differential equation which satisfies our criterion but for which the successive approximations do not converge. The statement is true but the proof that we give there does not make any sense. The argument should proceed as follows. We first prove that our criterion implies the following well known lemma.

Let $f(x,t)$ be a continuous function which is monotone non-increasing in x for all $0 \leqq t \leqq a$. Then the differential equation $dx/dt = f(x,t)$, $x(0) = 0$ has unique solution on $0 \leqq t \leqq a$.

The proof of this lemma is obtained by taking $V(x,t)$ to be x^2. Then (4) becomes

$$2(x-y)[f(x,t)-f(y,t)] \leqq \omega(V,t)$$

which is satisfied with $\omega = 0$. Since the Müller example satisfies the hypotheses of the lemma, we are done. It may be of some interest that the above criterion is a consequence of our theorem. It is not a consequence of Kamke's "allgemeine Eindeutigkeitssatz."

The results of Section 6, where we attempt to impose additional conditions to guarantee the convergence of successive approximations contain errors of a more serious nature. Our proof of Theorem 4 is incomplete, and it is not clear at the moment whether or not the theorem is true. The conclusion is correct if one imposes the additional hypothesis that V obeys the inequality $V(x+y, \max(t_1,t_2)) \leqq V(x,t_1) + V(x,t_2)$. Even so, the monotonicity condition on $\omega(r,t)$ seems to be indispensable.

We should like to thank Professors J. Dieudonné, P. Hartman, and J. P. LaSalle for calling these errors to our attention.

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* Received May 21, 1959.

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